

DYNAMICS AND STABILITY OF A WHEELSET/TRACK INTERACTION MODELLED AS NONLINEAR CONTINUOUS SYSTEM

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Abstract

There are many studies and various formulations of the problem dealing with self-excited oscillations of wheelset/rail system. In the majority of papers related to wheelset dynamics, the problem is investigated assuming that the angular velocity of wheels is equal to zero and the wheels are assumed as not deformable.

In this paper the results of the analysis high-frequency forced vibration of rolling wheelset interacting with rails by means of springs carrying the loading in three directions of relative displacement (vertical, lateral and longitudinal) and a spin of spring modelling rotational resistance are presented.

A wheelset is modelled by the system of two elastic wheels connected by an rigid axle. Wheel tyres are modelled as elastic curved Rayleigh beams with constant curvature, connected with the axle by continuous, visco-elastic foundation of Winkler type (forming wheel disc).

Introduction

The development of tracked transportation systems is promoted in several countries. With increasing travelling speed the dynamic interaction between train and track becomes very important. There is a need for simple but reliable models for wheel/rail systems in order to study the dynamic effects. The aim of this paper is to investigate dynamics of the model of wheel/ rail system.

The track is modelled as one or two infinite Bernoulli-Euler beams or Timoshenko beams on elastic or visco-elastic foundation, while the wheelsets are modelled as a various continuous or lumped subsystems. Both subsystems are in relative motion which is assumed of constant speed.

The solution of the problem is obtained applying the approach of travelling or standing waves. Particular attention is paid to the stationary solution and its stability. The

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determination of solution for Mathews problem (P.M. Mathews 1958) [1] solved for the Bernoulli-Euler or Timoshenko beam under moving harmonically oscillating load in [2] make it possible to investigate problem of stability of hybrid models [3].

Study for simple track models

Theoretical formulations which are intended to provide calculation models are generally limited to those influencing factors thought to be important. The particular significance of dynamic problems explains why increasing attention has been paid to the study of oscillation with the aid of theoretical calculation models which give a better insight into the phenomenon of corrugation formation.

The most significant factor is rail or wheel tyre vibration under the action of a moving and oscillating load. In an effort to find a solution, R. Bogacz et al. (1989) [2] examined the rail modelled as a Bernoulli-Euler beam or Timoshenko beam on an elastic foundation subjected to a moving and oscillating force.

To solve the problem, Mathews introduced a moving coordinate system connected with the force (Fig. 1) and expressed the response of the beam in the form of standing waves, which allows one to obtain the solution only in a region of small velocities and frequencies bounded by the curve of a „critical” solution (the first region in Fig. 2 - O, v_0, ω_0).

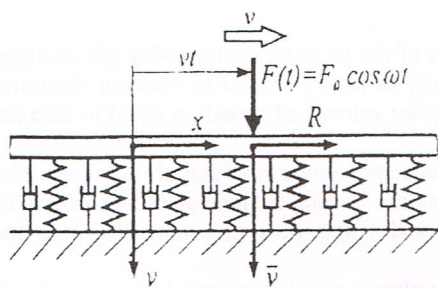


Fig. 1. System model with travelling load

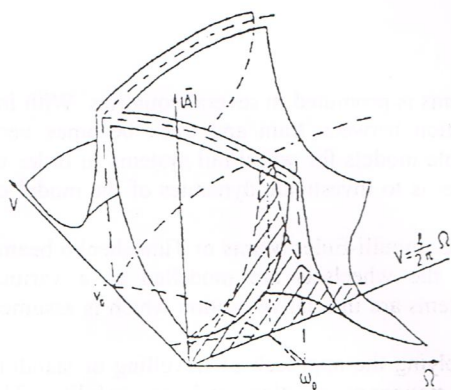


Fig. 2. Maximum displacement \bar{A} as function of velocity V and frequency Ω for Bernoulli-Euler beam

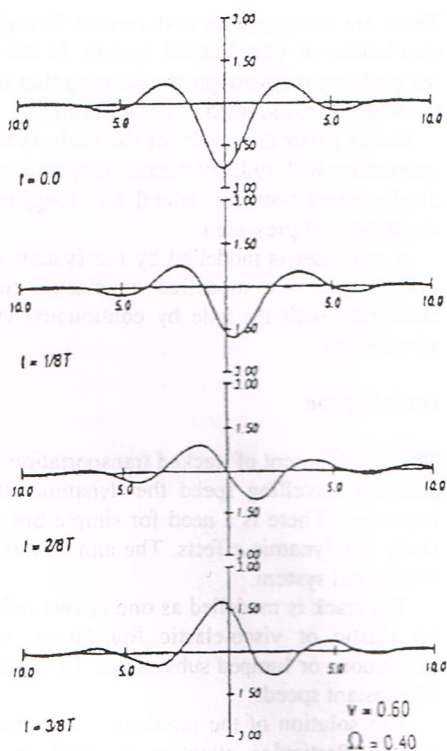


Fig. 3. Displacement of the beam for $t = 0, T/8, T/4, \text{ and } 3T/8$

S. Chonan [4] made a similar investigation of Timoshenko beam under a moving harmonically oscillating load. Like Mathews, he also assumed the beam displacement in the form of standing waves and consequently could not find a general solution for the whole speed-frequency range. An alternative approach done in [2], which yields a solution in the form of travelling waves, allows one to estimate the displacements (and stresses) in the high-frequency range.

The equations of the Timoshenko beam motion regarding the effects of shear deformation and rotary inertia are given as follows:

$$\begin{aligned} EI \frac{\partial^2 w}{\partial x^2} + \kappa AG \left(\frac{\partial w}{\partial x} - \psi \right) \rho A \frac{\partial^2 \psi}{\partial t^2} &= 0, \\ \kappa AG \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x} \right) - \rho A \frac{\partial^2 w}{\partial t^2} - b \frac{\partial w}{\partial t} - cw &= F_0 \delta(x - vt) e^{i\omega t}. \end{aligned} \quad (1)$$

We introduce dimensionless quantities V and Ω defined by:

$$\omega_0^2 = \frac{c}{\rho A}, \quad V = \frac{v}{v_0}, \quad \Omega = \frac{\omega}{\omega_0}, \quad v_0 = \sqrt{\frac{cI}{\rho A^2}}, \quad (2)$$

here A is the beam (rail) cross-section area, c is the elastic compliance modulus of the foundation, E is Young's modulus, I moment of inertia of the cross-section area, κ coefficient describing the effective shear area, and ρ is the mass density of the beam.

Using the Fourier transform technique we obtain a set of linear algebraic equations.

Let us now consider the particular case of an pure elastic beam. We introduced a coordinate system moving with the same velocity as the oscillating force.

The solution of Eqs. (1) can be written in the form

$$W(R_0, \tau) = W_1(R_0) \cos \Omega \tau + W_2(R_0) \sin \Omega \tau, \quad (3)$$

where:

$$W = w \frac{8EIa_0^3}{F_0}, \quad a_0 = \sqrt{\frac{c}{4EI}}, \quad R_0 = a_0 x - v\tau, \quad \tau = \omega_0 t.$$

Substituting (3) into equation of motion (1) we obtain two ordinary differential equations. The roots of the characteristic equation may be complex or all real or two may be real and two complex, depending on the values of the coefficients of the polynomial.

According to the range of the load velocity V , corresponding solutions for displacement and rotation differ in the number of waves ahead of the load and behind the load.

The discussion essential differences between the solutions for various values of velocity V and frequency Ω are given in [2].

As an example the displacements for the times $t = 0, T/8, T/4, 3T/8$ are shown in Fig. 7.

Physical and mathematical model of the wheel

The wheel tyre is modelled by an elastic curved Rayleigh beam connected to the axle by means of continuous elastic Winkler-type foundation. The elastic foundation constituting the

wheel disc enables to transverse the loads in three directions: circumferential, radial and vertical to the plane of a wheel. The use of curved beams theory causes the preservation of the real shape of the tyre cross-section. Visco-elastic properties of the wheel material are describe by the Kelvin-Voigt's model:

$$q_j = - \left(k_j u_{oj} + c_j \frac{\partial u_{oj}}{\partial t} \right), \quad (4)$$

where

q_j, u_{oj} ($j=1,2,3$) - elastic foundation reactions and displacements in circumferential, radial and transversal directions, respectively,

k_j, c_j - elastic foundation stiffness and damping.

The following coordinate system are assumed to describe the 3-dimensional mathematical model of rotating railway wheel (Fig.4):

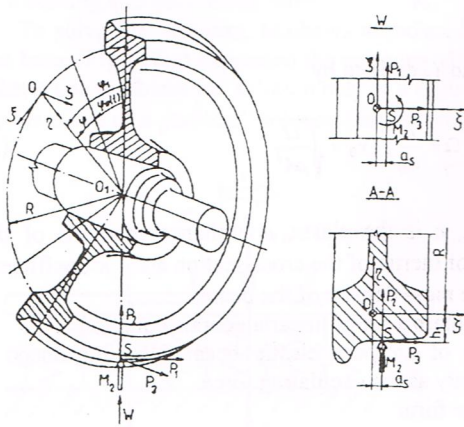


Fig.4. Coordinate systems and exciting forces

-polar system φ, R with a pole in the wheel centre, rigidly connected with the rotating wheel; by means of there coordinates the geometrical axis of the tyre has been described,

-polar system φ_1, R with the pole in the wheel centre, using to the description of the rotation motion of the wheel,

-rectangular, dexterorotatory system of coordinates ξ, η, ζ with origin O on geometrical axis of tyre and a position given by spatial coordinate φ or φ_1 ; axis ξ, η, ζ constitute tangential, normal and binormal to the undeformed axis of a wheel; this system of coordinates is used to describe displacements, internal and external forces and cross-section of wheel tyre.

The geometrical axis of wheel tyre has been defined by a locus of geometrical centres of gravity of cross-sections undeformed wheel tyre. Assuming the angular velocity of the wheel $\dot{\varphi}_o = \text{const}$ the relation between φ, R and φ_1, R takes the form:

$$\varphi_1 = \varphi + \dot{\varphi}_o t. \quad (5)$$

The problem is now more complicated due to curved beam and much more dimensions. Detailed examinations of the problem in case of two dimension were carried out in [5].

The system of coupled differential equations which describe forced vibrations of the wheel tyre rotating at the velocity $\dot{\varphi}_o$, including visco-elastic properties can be written in polar coordinates φ, R in the form:

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \left[\frac{\partial}{\partial \varphi} (-m_1 v + m_2 w) + m_3 u \right] + \frac{\partial}{\partial t} \left[\frac{\partial}{\partial \varphi} \left(-a_1 \frac{\partial u}{\partial \varphi} + a_2 v + a_3 w \right) + a_4 u - 2\dot{\varphi}_o (m_4 v + s_1 \vartheta) \right] + \\ + \frac{\partial}{\partial \varphi} \left(-k_1 \frac{\partial u}{\partial \varphi} + k_2 v - k_3 w \right) + k_4 u = R q_\varphi + m_\zeta, \end{aligned}$$

$$\begin{aligned} & \frac{\partial^2}{\partial t^2} \left[\frac{\partial^2}{\partial \varphi^2} (m_5 v + m_6 w) - m_7 \frac{\partial u}{\partial \varphi} - m_8 v - s_2 \vartheta \right] + \frac{\partial}{\partial t} \left[-a_5 \frac{\partial^4 v}{\partial \varphi^4} + \frac{\partial^2}{\partial \varphi^2} [(-2a_5 + a_6)v + a_7 w] + \right. \\ & + a_2 \frac{\partial u}{\partial \varphi} - (a_5 + a_8)v + d_1 \vartheta + 2\dot{\varphi}_0 \left[\frac{\partial}{\partial \varphi} (m_5 v - m_{11} w - s_3 \vartheta) - m_{11} u \right] \left. - k_5 \left(\frac{\partial^4 v}{\partial \varphi^4} + 2 \frac{\partial^2 v}{\partial \varphi^2} + v \right) - \right. \\ & \left. - \frac{\partial^2}{\partial \varphi^2} \left(k_6 \frac{\partial^2 w}{\partial \varphi^2} - k_7 v - k_8 w - t_1 \vartheta \right) + k_2 \frac{\partial u}{\partial \varphi} - k_9 v + t_2 \vartheta = -Rq_\eta + \frac{\partial m_\zeta}{\partial \varphi} + s_8 \dot{\varphi}_0^2 \right. \end{aligned} \quad (6)$$

$$\begin{aligned} & \frac{\partial^2}{\partial t^2} \left[\frac{\partial^2}{\partial \varphi^2} (m_{12} v + m_{13} w) + m_{14} \frac{\partial u}{\partial \varphi} - m_{15} w - s_4 \vartheta \right] + \frac{\partial}{\partial t} \left[-a_9 \frac{\partial^4 w}{\partial \varphi^4} + \frac{\partial^2}{\partial \varphi^2} (a_7 v + a_{10} w + d_2 \vartheta) - \right. \\ & - a_3 \frac{\partial u}{\partial \varphi} - a_{11} w - d_3 \vartheta - 2\dot{\varphi}_0 \frac{\partial}{\partial \varphi} (m_{16} v + s_5 \vartheta) \left. - \frac{\partial^4}{\partial \varphi^4} (k_6 v + k_{10} w) + \frac{\partial^2}{\partial \varphi^2} (k_8 v + k_{11} w + t_3 \vartheta) - \right. \\ & \left. - k_3 \frac{\partial u}{\partial \varphi} - k_{12} w - t_4 \vartheta = -Rq_\zeta + \frac{\partial m_\eta}{\partial \varphi} \right. \end{aligned}$$

$$\begin{aligned} & \frac{\partial^2}{\partial t^2} (m_{17} v + m_{18} w + s_6 \vartheta) + \frac{\partial}{\partial t} \left\{ -\frac{\partial^2}{\partial \varphi^2} (a_{12} w + d_4 \vartheta) - a_{13} v + a_{14} w + d_5 \vartheta + 2\dot{\varphi}_0 \left[\frac{\partial}{\partial \varphi} (m_{12} v + \right. \right. \\ & \left. \left. + m_{13} w) + m_{14} u \right] - \frac{\partial^2}{\partial \varphi^2} (k_6 v + k_{13} w + t_5 \vartheta) - k_{14} v + k_{15} w + t_6 \vartheta = m_\xi - s_9 \dot{\varphi}_0^2 \right\} \end{aligned}$$

where:

u, v, w - point O displacements along axes ξ, η, ζ (Fig.4),

ϑ - rotation angle of the wheel tyre cross-section in relation to ξ direction,

m_i - reduced masses of the wheel tyre and disc,

s_i - reduced mass moments of the first order,

k_i, t_i - wheel tyre and disc reduced stiffnesses,

a_i, d_i - damping equivalent coefficients of the wheel tyre and disc,

$q_\xi, q_\eta, q_\zeta, m_\xi, m_\eta, m_\zeta$ - external forces and moments distributed continuously along geometrical axis of wheel tyre.

The system of equations (6) is the mathematical 3-dimensional model of the rotating at the angular velocity $\dot{\varphi}_0$ wheel. The first two equations refer to the motion of the wheel tyre in its plane (circumferential vibrations and flexural radial), when the remaining two describe the motion out the plane (flexural and torsional vibrations). The vibrations in the wheel plane and vibrations out of the wheel plane are coupled by means of elastic and inertial forces.

Vibrations are excited by harmonic concentrated forces acting at the contact point S . Spin moment M_2 has been also taken into account as a source of excitation. The positive senses of exciting force have been assumed according to the senses of axis ξ, η, ζ (Fig.4).

The solution of the system of equations (6) describing forced vibration of the rotating wheel is sought for in the co-ordinates system φ, R in the form:

$$u(\varphi, t) = \frac{1}{2\pi} T_{10}(t) + \frac{1}{\pi} \sum_{n=1}^{\infty} [T_{11}(t) \cos n\varphi_1 + T_{12}(t) \sin n\varphi_1] \quad (7)$$

The amplitude A_u of vibration of point O can be expressed as follows:

$$A_u(\varphi_1) = \frac{1}{\pi} \left\{ \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_{1n} \cos n\varphi_1 + a_{2n} \sin n\varphi_1) \right]^2 + \left[\frac{b_0}{2} + \sum_{n=1}^{\infty} (b_{1n} \cos n\varphi_1 + b_{2n} \sin n\varphi_1) \right]^2 \right\}^{\frac{1}{2}}. \quad (8)$$

Similarly the displacements v , w , ϑ and the amplitudes A_v , A_w , A_ϑ

Numerical results

Figure 5 present in the form of frequency response functions the results of the numerical calculations for forced vibrations of the railway wheel. Characteristics refer to the point O for coordinate $\varphi_1 = \pi$. Amplitudes $A_u(\pi)$, $A_v(\pi)$, $A_w(\pi)$ and $A_\vartheta(\pi)$ have been determined for the first eleven modes of vibrations and their values have been given in dB assuming reference level 10^{-11} m. Separate cases concerning the following types of vibrations:

- case „a” - circumferential vibrations $A_u(\pi)$,
- case „b” - flexural radial vibrations $A_v(\pi)$,
- case „c” - flexural vibrations out of the wheel plane $A_w(\pi)$,
- case „d” - torsional vibrations wheel tyre $A_\vartheta(\pi)$.

The numerical analysis has been performed for the nominal wheel diameter of 0.95 m, for the angular velocities $\dot{\varphi}_0$ of the wheel, which correspond to the linear velocities 0, 200, 400 km/h in the rolling motion.

The another results of the numerical calculations can find in [6]. It is interesting that for velocity about 200 km/h the amplitude of vibration with frequency about 100 Hz is many times larger than for the velocity equal to zero or velocity about 400 km/h.

Model of wheelset

The physical model of wheelset is constituted by two elastic wheels, connected by the rigid axle. The wheels are coupled with rails by linear Hertz springs, which transfer the forces in three directions. But the moment of spin is assumed as a nonlinear function of normal force. The wheelset rolls within the track without the creepage at the constant velocity.

The motion of wheelset is described by a set of six differential equations. The first three of them describe the motion of its gravity centre, whereas the next three ones describe the rotation about this fixed centre. The equations of motion of centre of mass are written in the system x , y , z , whereas the equations of rotation about the centre, written in the system of main central axes of inertia of wheelset x_2 , y_2 , z_2 . Assuming $\dot{\varphi}_0 = \text{const}$ and omitting the nonlinear members except dependence normal spring stiffness - stiffness of spin (Fig. 7) model of the wheelset is obtained. The linearized model is used to investigate the rolling motion stability while adequate linearization allows to determine the limit cycles [8].

The vibrations of wheelset for the assumed model may be divided into simple modes, depending on symmetry or antymetry of deformation of the wheels.

The critical values of V which result in the boundaries of the instability region S_I are determined in the frequency-velocity plane by the straight lines $V = V_{cr}$ tangent to the curves obtained from condition of existence of nontrivial solution. In the range $S_I = \{V: V \in [V_{1cr}, V_{2cr}]\}$ cf. Fig.6 the solution describes waves which propagate in the wheel tyre with amplitudes

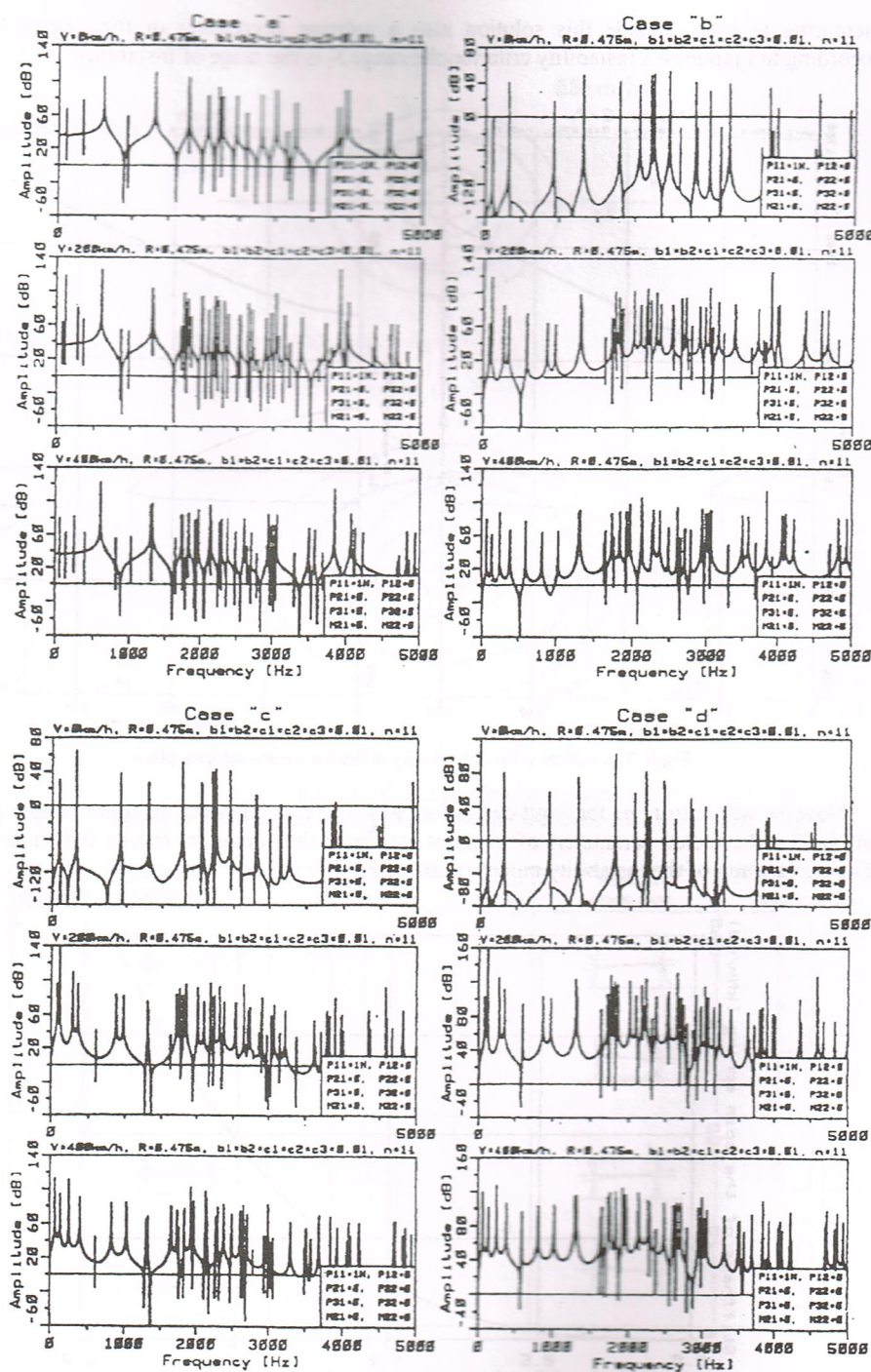


Fig.5: Frequency response functions for a rotating railway wheel

increasing in time. Beside this solution also a solution decreasing in time exists, thus, according to Ljapunov's instability criterion the range S_I is the range of instability.

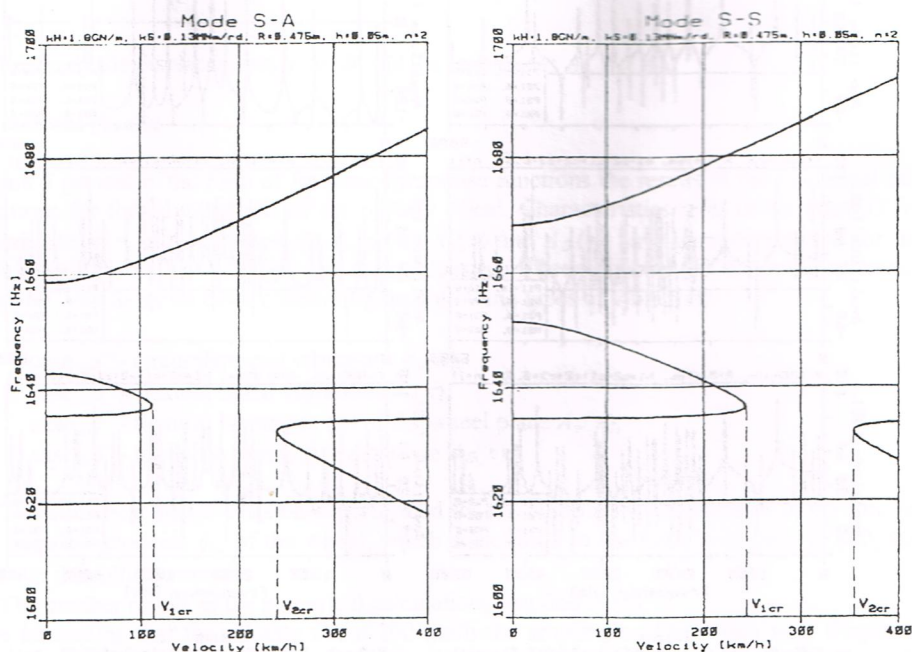


Fig. 6: The critical values of velocity in the frequency-velocity plane

Now we will determine the limit cycles for particular cases. For the moment of spin plotted in Fig. 7 and selected parameters of wheelset we obtain the numerical results shown in Fig. 8. The explanation of the instability modes illustrate Fig. 9.

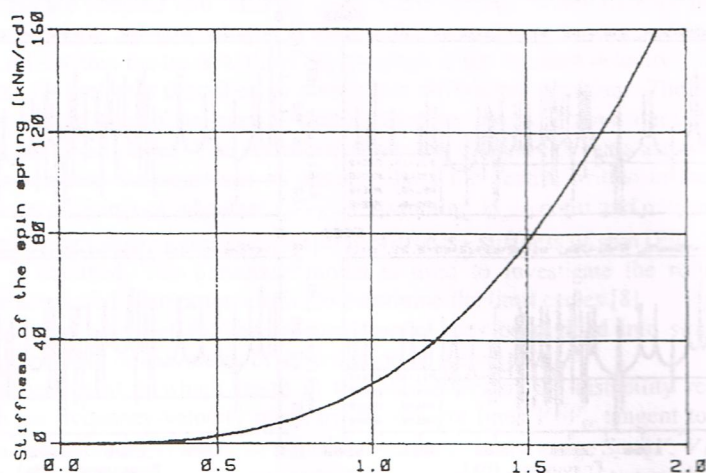


Fig. 7: Stiffness of the spin moment versus stiffness of Hertz spring

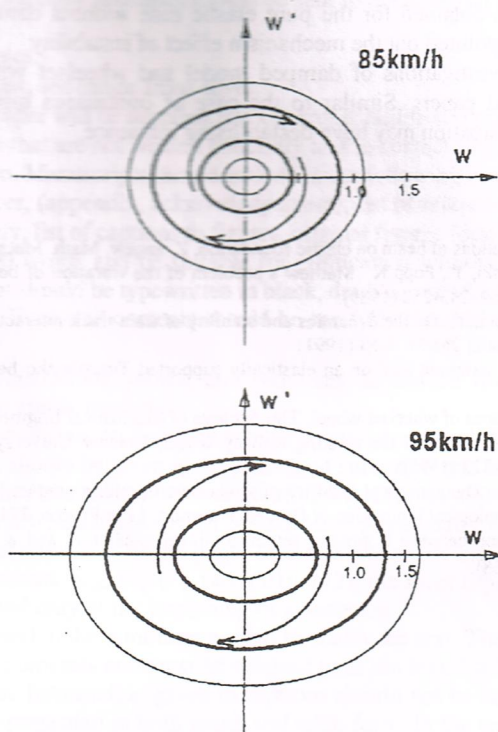


Fig.8. Stable and unstable limit cycles

Ranges of instability: for 85 km/h ($0,4 \div 0,7$ mm) and for 95 km/h ($0,69 \div 1,35$ mm)

Mode of vibrations	Motion in vertical plane	Motion in horizontal plane	Mode of vibrations	Motion in vertical plane	Motion in horizontal plane
S - S			A - S		
S - A			A - A		

Fig.9: The instability modes of the wheelset

The above results obtained for the pure elastic case without damping have an academic importance only but pointed out the mechanism effect of instability.

The results of investigations of damped model and wheelset with flexible axle will be presented in the next papers. Similar to the case of continuous model of train-track interaction the energy dissipation may have destabilising influence.

References

- [1] Mathews, P.M.: Vibrations of beam on elastic foundation, *Z. angew. Math. Mech.* 38, 105-115 (1985).
- [2] Bogacz, R., Krzyżyński, T., Popp K.: Mathew's problem of the vibration of beam on elastic foundation, *Z. angew. Math. Mech.* 69, 243-252 (1989).
- [3] Bogacz, R., Krzyżyński, T.: On the dynamics and stability of train-track interaction, *Czasopismo Techniczne Politechniki Krakowskiej* 2M/91, 3-53 (1991).
- [4] Chonan, S.: Moving harmonic load on an elastically supported Timoshenko beam, *Z. angew. Math. Mech.* 58, 9-15 (1978).
- [5] Dżuła, S.: Free vibrations of wheelset wheel, *The Archive of Mechanical Engineering* 2/3, 97-124 (1989).
- [6] Dżuła, S.: Forced vibrations of the rotating railway wheel, *Cracow University of Technology, Selected Problems Vol. 3*, 307-323 (1995).
- [7] Bogacz, R., Dżuła, S.: Dynamics and stability of a wheelset in rolling contact motion on rails, *International Symposium on Technological Innovation in Guided Transport, Lille-France*, 871-883 (1993).
- [8] Bogacz, R.: Interaction between a moving set of nonlinear oscillators and a travelling wave, *Proc. Vibr. Probl.* 1,9, 35-57 (1968).