

ON AN ANALOGY BETWEEN ELASTIC INSTABILITY AND NUCLEAR EXPLOSIVES

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Abstract

An intrusive analogy is developed between two subjects of considerable applied importance in civil engineering namely buckling instability of the Euler elastica and critical mass for neutrons diffusion.

Introduction

Vibrational resonance as well as elastic buckling of engineering structures are fairly developed and well understood subjects within applied mechanics [1]. It is therefore quite useful to establish and discuss some analogies which do exist between this field and another seemingly very remote field related to determining the critical mass of certain active materials which is required for a sustained fission in a thermal or fast neutrons diffusion. To develop such an analogy between two problems which are considerable applied importance in engineering and applied physics is the aim of the present note.

I. Part One – Axial load

1. The elastica

Determining the so called buckling loads of thinwalled elastic structures which may become unstable under the action of external or axial pressure is a very important theoretical problem with numerous applications in science and technology [1]. In particular, the so called Euler elastica was used extensively in illustrating fundamental aspects of Poincare and Hopf symmetry breaking bifurcation, Rene Thom's classification theorem of elementary catastrophes and more recently spacial, purely statical, deterministic chaos [1-3].

The exact nonlinear differential equation of the Euler elastica is easily derived and may be written in terms of the rotation angle $\phi(x)$ or the lateral deflection $\omega(x)$ as [1]

$$\phi'' + \lambda^2 \sin \phi = 0, \quad (1a)$$

or

$$\frac{\omega''}{(1-\omega^2)^{1/2}} + \lambda^2 \omega = 0, \quad (1b)$$

respectively where: $\lambda = \sqrt{P/EI}$, P is the axial compression force, EI is the bending stiffness, $(\prime) = d(\)/dx$ and x is the axial coordinate. Confining our investigation to determining the eigenvalue critical load of Eqs. (1 a) and (1 b), we are permitted to linearize and find that

$$\phi'' + \lambda^2 \phi = 0, \quad (2a)$$

$$\omega'' + \lambda^2 \omega = 0. \quad (2b)$$

Assuming the boundary conditions to be that of a simple support [1], the eigenvector of the above equation is easily found from solving Eq.(2a) to be

$$\omega = a \sin \frac{i\pi}{l} x, \quad (3)$$

where $a = w(x = l/2)$, i is the wave number and l is the length of the elastica. The associated eigenvalue is easily determined from inserting Eq.(3) into Eq.(2a)

$$a \left(\frac{i\pi}{l} \right)^2 \sin \frac{i\pi}{l} x - a \lambda^2 \sin \frac{i\pi}{l} x = 0. \quad (4)$$

The eigenvalue critical load is thus

$$\lambda_c^2 = \frac{P_c}{EI} = \frac{i^2 \pi^2}{l^2}. \quad (5)$$

The smallest eigenvalue is obviously associated with $i = 1$ which gives the well known Euler buckling load [1]

$$P_c = \frac{EI\pi^2}{l^2} \quad (6)$$

of a simply supported strut.

Needless to mention that different boundary conditions give different buckling loads and buckling forms, i.e. eigenvectors as discussed in great detail in [1].

Next we look at the analogous problem of determining the critical mass of neutrons diffusion.

2. The diffusion of neutrons

It is well known that neutrons diffusion in fissile material is governed by a so called Helmholtz partial differential equation [4]. It is further noted that it is immaterial whether we seek a solution for neutrons density $N = N(xt)$ or neutrons flux $\phi = \phi(xt)$ where t is the time because both of the two differential equations have the same mathematical form namely

$$\frac{1}{V} \frac{\partial N}{\partial t} = (\nu \Sigma_f - \Sigma_a) N + D \nabla^2 N \quad (7a)$$

, and

$$\frac{1}{V} \frac{\partial \phi}{\partial t} = (\nu \Sigma_f - \Sigma_a) \phi + D \nabla^2 \phi, \quad (7b)$$

respectively. Here Σ_f and Σ_a are the macroscopic cross-sections of the active material for fission and absorption, respectively. ν is the number of neutrons released at each fission, V is the average velocity of the neutrons and $\nabla^2 = \nabla \cdot \nabla$ is the Laplacian operator [4-6]. The last two equations (Eqs. (7a) and (7b)) cannot be compared to those of the elastica (2) because they are partial differential equations dependent on space (x) and time (t) while the elastica is governed by an ordinary time independent differential equation.

It must therefore be regarded as a rather fortunate situation that Eqs. (7a) and (7b) turned out to be stationary for the physically most important case namely the so called criticality condition. This critical state is vital for the practical situation of designing a self sustained fission reaction for extracting energy. For self sustained fission our equation must be stationary, that means time independent. Physically this would mean that a state is maintained where neutrons production is exactly balanced by neutron losses due to leakage and absorption. Mathematically however our equations become ordinary differential equations which have the same form as (2). In fact in the one-dimensional case, they are identical to the elastica eigenvalue equation. The diffusion equation may be then written as [4-6]

$$\nabla^2 N(x) = \left(\frac{\nu \Sigma_f - \Sigma_a}{D} \right) N(x) = 0, \quad (8)$$

where ∇^2 is now simply $\partial^2/\partial x^2$ and D is the diffusion constant. Consequently and noting that constant neutron velocity

$$\frac{1}{V} \frac{\partial}{\partial t} N(X, t) = 0 \quad (9)$$

leads to

$$\frac{\partial N(x, t)}{\partial t} = 0. \quad (10)$$

We have the two alternative forms of the Helmholtz equation [4]

$$N'' + B^2 N = 0 \quad (11)$$

and

$$\phi'' + B^2 \phi = 0, \quad (12)$$

where

$$B^2 = \frac{\nu \Sigma_f - \Sigma_a}{D}$$

We note that Eqs. (11) and (12) have been studied extensively in the mathematical literature as well as in applied physics, engineering vibration and elastic stability [1].

Before proceeding to the next stage which is to solve the diffusion equation for the corresponding geometry and material composition, we note that the quantity

$$B^2 = \frac{\nu \Sigma_f - \Sigma_a}{D} \quad (13)$$

is known in reactor design theory as the material buckling of the reactor although we are not aware of any explanation given for the use of this expression. However, now and in view of the formal identity between the critical neutrons diffusion equation and that of the linearized elastica equation, this expression is quite understandable. Nevertheless it is misleading because a reactor does not buckle. The quantity B^2 is at best a reactor critical value which is analogous to the critical eigenvalue buckling load of the Euler Elastica or the eigenfrequency of a vibrating structures. It is therefore a far better terminology to call B^2 the critical eigenvalue of the reaction and the corresponding solution, the eigenmode or eigenvector of the reaction at criticality.

3. The effect of geometry

For basic simple geometries and boundary conditions, the "reactor" diffusion at criticality could be just as easily solved as any classical elastic buckling problem [1]. Ignoring boundary conditions, it is evident that a sine or cosine function shows that

$$B^2 = \left(\frac{\pi}{\bar{l}} \right)^2$$

where \bar{l} is a geometric quantity. This result confirms the obvious intuitive feeling that for certain fissile material at criticality there is only one particular geometry corresponding to it. For the trivial case of a one-dimensional reactor in the form of an infinitely long slab, the solution is

$$N = C_1 \cos \frac{\pi x}{\tilde{a}}, \quad (14)$$

where C_1 is a constant \tilde{a} is the extrapolated thickness of the physical thickness of the slab. Consequently inserting Eq. (14) in Eq. (11) one find that

$$B^2 = \left(\frac{\pi}{\tilde{a}} \right)^2. \quad (15)$$

This way the thickness of the slab for which neutrons absorbtion and neutrons leakage are balanced is determined. The classical well known cases for the other fundamental geometrical shapes namely the sphere, the rectangular and the cylindrical reactor give the critical eigenvalues (the so called geometrical buckling)

$$B_s^2 = \left(\frac{\pi}{R} \right)^2 \quad (16)$$

$$B_r^2 = \left(\frac{\pi}{\tilde{a}} \right)^2 + \left(\frac{\pi}{\tilde{b}} \right)^2 + \left(\frac{\pi}{\tilde{c}} \right)^2 \quad (17)$$

and

$$B_{cy}^2 = \left(\frac{2.405}{\tilde{R}} \right)^2 + \left(\frac{\pi}{\tilde{H}} \right)^2, \quad (18)$$

respectively where \tilde{R} is the radius of the sphere, $\tilde{a}, \tilde{b}, \tilde{c}$ are the dimensions the rectangular and \tilde{H} is the height of the cylinder. This corresponds to selfweight buckling.

At this point we should address the question of the role played by geometry and the most optimal geometrical shape which would clearly correspond to the smallest critical mass for the onset of steady state fission. This question can be answered without involving material composition which can be eliminated from the analysis. For simplicity we start by comparing the spherical shape with that of a cubic shape ($\tilde{a} = \tilde{b} = \tilde{c}$). Now at criticality we must have

$$B_s^2 = B_r^2 \quad (19)$$

and therefore

$$\left(\frac{\pi}{\tilde{R}} \right)^2 = 3 \left(\frac{\pi}{\tilde{a}} \right)^2. \quad (20)$$

Consequently one finds that

$$\tilde{a} = \tilde{R}\sqrt{3}. \quad (21)$$

Since the critical mass of a sphere and a cube made of certain fissile material are

$$m_s = V_s \rho = \left(\frac{4}{3} \pi R^3 \right) \rho \quad (22)$$

and

$$m_r = V_r \rho = \left(\frac{4}{3} \pi R^3 \right) \rho, \quad (23)$$

respectively, then one finds that

$$\frac{m_r}{m_s} = \frac{V_r \rho}{V_s \rho} = \frac{9\sqrt{3}}{4\pi} = 1.24049. \quad (24)$$

In other words, the critical mass of a cubic reactor is about 24% larger than that of a sphere. Next we would like to investigate the efficiency of the cylinder compared to the sphere. To do that we have to find first the optimal cylindrical form.

From the critical eigenvalue of the cylinder as given by Eq. (18) and solving for \tilde{R}^2 one finds

$$\tilde{R}^2 = \frac{(2.405)^2}{B_{cy}^2 - \left(\pi / \tilde{H} \right)^2}. \quad (25)$$

The corresponding volume of the cylinder is then

$$V_{cy} = \pi \left(\frac{2.405}{B_{cy}^2 - (\pi / \tilde{H})^2} \right) \tilde{H}. \quad (26)$$

The minimum of V_{cy} is given by

$$\frac{dV_{cy}}{d\tilde{H}} \quad (27)$$

and leads to

$$\tilde{H}_0 = \frac{\pi\sqrt{3}}{B_{cy}}. \quad (28)$$

Inserting in \tilde{R}^2 one finds

$$\tilde{R}_0 = \frac{2.405}{\pi\sqrt{2}} \tilde{H}_0 = 0.54131 \quad (29)$$

The critical eigenvalue of the optimal cylinder is thus

$$B_{cy}^2 = \frac{8.676037}{\tilde{R}_{cy}^2}. \quad (30)$$

Setting $B_{cy}^2 = B_s^2$ one finds that

$$\frac{8.676037}{\tilde{R}_{cy}^2} = \frac{\pi^2}{\tilde{R}_s^2}.$$

From which one obtain

$$\tilde{R}_{cy} = \tilde{R}_s (0.937585403). \quad (32)$$

Consequently

$$\frac{m_{cy}}{m_s} = 1.4194073 \quad (33)$$

That means the critical mass of the optimal cylinder is only 14% larger than that of the sphere.

4. The numerical value of the critical mans

So far our analysis has been confined to finding general expressions. To obtain numerical values for the critical mass we need to consider the properties of the fissile material. The measurable critical dimensions of the reactor are thus obtained from equating the critical value B^2

expressed in terms of the geometry of the reactor, the so called "geometrical buckling" to that of B^2 expressed in terms of the cross-sections and other physical parameters of the material of the reactor, the so called "material buckling" which we must determine next.

The first material quantity we will determine here is the diffusion coefficient D . A simple expression may be found for D using the so called one speed approximation to transport theory. In this approximation D can be written in terms of the macroscopic transport cross section as [4-6]

$$D = \frac{1}{3\Sigma_t}, \quad (34)$$

where Σ_t is the macroscopic transport cross section. It is also some times more convenient to work with the so called diffusion length which is defined as

$$L = \sqrt{\frac{1}{(3\Sigma_t\Sigma_a)}}, \quad (35)$$

where Σ_a is the absorptions cross-section.

Further, we may recall from the elementary theory of neutrons diffusion that first, the macroscopic cross-section Σ_0 is related to microscopic cross section σ_0 by the relation

$$\Sigma_{(\dots)} = \bar{N}\sigma_{(\dots)}, \quad (36)$$

where \bar{N} is the nuclei number and second that the mean free path is given by

$$l_{(\dots)} = \frac{1}{\Sigma_{(\dots)}}. \quad (37)$$

Taking as a fissile material U_{92}^{235} and considering fast rather than thermal neutrons, then we may use the following experimentally determined values¹ which are of course subject to the usual experimental scatter: $\bar{N} = 0.0482(10)^{24}$ nuclei per cm^3 ; $n = \bar{N}/\rho = A/A_0\rho = 2.5368(10)^{21}$; $\sigma_t = 4$ barn, $\sigma_f = 1.5$ barn; $\sigma_a = 1.65$ barn and $\nu = 2.4$, where ν is the neutrons number per fission. Evaluating for these values one find that [4-6]

$$\Sigma_f = \bar{N}\sigma_f = (7.23)(10)^{-2}, \quad (38)$$

$$\Sigma_a = \bar{N}\sigma_a = (7.953)(10)^{-2}, \quad (39)$$

and

$$\Sigma_t = \bar{N}\sigma_t = (19.28)(10)^{-2}. \quad (40)$$

¹ The relation between the microscopic cross-sections are given as following: the total is $\sigma_{t0} = \sigma_s + \sigma_a$, where σ_s means scattering and σ_a means absorbtion. We have further $\sigma_a = \sigma_c + \sigma_f$, $\eta = \nu/(1+\alpha) = \nu(\sigma_f/\sigma_a)$, and $\sigma_{t0} = 1.06(A)^{2/3}(10)^{-29}$ m²/nuclous i.e. $\sigma_{t0} = 0.106A^{2/3}$ barn. Finally $\Sigma_t = \Sigma_{t0} - \Sigma_s\bar{\mu}_0$, where $\bar{\mu}_0 = (2/3)A$. Note that for isotropic scattering $\bar{\mu}_0 = 0$ and $\Sigma_t = \Sigma_{t0}$. For heavy nuclei the factor $[1/(1-\bar{\mu}_0)] \rightarrow 1$ and thus we have $\Sigma_s \approx \Sigma_t$.

Inserting in L , D and B one finds

$$L = \sqrt{\frac{1}{(3\Sigma_t \Sigma_a)}} = 4.6625, \quad (41)$$

$$D = L^2 \Sigma_a = \frac{1}{3\Sigma_t} = 1.728907 \quad (42)$$

and

$$B = \sqrt{\frac{\nu\Sigma_f - \Sigma_a}{D}} = \sqrt{(5.436)(10)^{-2}} = 2.3316(10)^{-1}. \quad (43)$$

Therefore the critical radius is

$$R_c = \frac{\pi}{B} = 13.474 \text{ cm} \quad (44)$$

and the corresponding critical mass using $S=19.05 \text{ g/cm}^3$ of metallic U^{235} is

$$M_c = 195.2 \approx 200 \text{ kg}. \quad (45)$$

This is identical to the classical result obtained for the first time by Serber [7] a long time ago.² This result over estimates the critical radius by a considerable amount because we have used an elementary diffusion theory not taking the true geometrical boundary conditions into account. We have been effectively using a theory developed for thermal neutrons to calculate M_c for fast neutrons which is quite different from the thermal case.

Nevertheless the main conceptual thinking behind the theory is sufficiently transparent when using the present simple analysis so that we can easily propose some refinements of the calculations which we do next. To improve the preceding estimate $M \approx 200 \text{ kg}$ we present two methods. Let us give first the simplest of the two methods. It is based on an initially crude estimation of the mean free path of transport l_t . We know from Lawson criterion of fusion that

$$Nt \geq 10^{14} \text{ s/cm}. \quad (46)$$

Consequently the time t must be of the order

$$t \geq \frac{(10)^{14}}{[(0.0482)(10)^{22}]} \approx 1.21(10)^{-8} \text{ s}. \quad (47)$$

² One could argue that the diffusion equation used her is equivalent to a classical Brownian motion where the dimension of a particle path is $d=1$. On the otherhand a quantum object with $1/2$ spin like the fission neutron must have according to Σ^∞ space theory a fractal Hausodovff 'dimension' $d_c^{(0)} = 2/3$. = $2/3$. For the four-dimensional Σ^∞ space we must take a factor $(2/2)^{(4-1)} = (2/3)^3$ into account. This leads to a critical mass $M_c = (2/3)^3(195.2) = 57.83 \text{ kg}$ which agrees with the improved value mentioned in a 1997 book edited by King, namely $M_c=56 \text{ kg}$ for U^{235} in pure metallic form.

Since

$$l \geq tV,$$

where $V \approx 2(10)^9$, cm/s is the velocity of fast fission neutrons, then we find that

$$l \geq (0.21)(2)(10) = 4.2 \text{ cm.} \quad (48)$$

Therefore we can assume that the mean free path of fission is $l \approx 5$ cm.

To see that this is quite reasonable we calculate the mean transport free path for U^{235} and find that³

$$l_t = \frac{1}{\Sigma_t} = \frac{1}{0.1905} \approx 5.25 \text{ cm.} \quad (49)$$

In agreement with the experimental evidences it is reasonable to assume that⁴

$$R \geq (1/0.8)l \geq (1/0.8)(5.25) \approx 6.6 \text{ cm.} \quad (50)$$

The reduction factor is thus

$$\frac{\tilde{R}}{R} \approx \frac{13.5}{6.6} \approx 2. \quad (51)$$

Therefore the corresponding critical mass is found from Eq.(45) and Eq.(51) to be

$$M_c \approx \frac{200}{(2)^3} = 25 \text{ kg.} \quad (52)$$

It is just a coincidence that this value is exactly the same value found when applying R . Oppeneimer's improvement of R . Serber's result using a neutrons diffusion theory which include a neutron reflector for the fast reaction [7].

5. The effect of neutron reflector

The second method to check and improve Serber's classical result $M \approx 200$ kg is based on an improved diffusion theory formula [7]. The originally improved theory used here is due to Oppenheimer, however it was written in a somewhat cumbersome form and in addition Ref. [7] contains printing errors so that for the sake of clarity we rederive it in a simpler way starting from our more familiar expression for the so called geometrical buckling

$$B^2 = \frac{\nu\Sigma_f - \Sigma_a}{D}. \quad (53)$$

³ This is an estimation based on the limiting case of neutrons energy in excess of a few MeV, namely $\sigma_t \approx 2\pi R_n^2$ and approximate value for R_n , the radius of the nuclei of light mass number $R_n \approx (1.3)(10)^{-15} A^{1/3}$ meter which gives for U^{235} the value $\Sigma_t = 20/\text{meter}$ and a mean free path $l = 1/\Sigma_t \approx 50$ mm.

⁴ It is frequently reasoned that $R_c > l_t$ is very restrictive assumption and that $R_c \leq l_t$ is sufficient for fast fission

Noting that

$$D = \frac{1}{3\Sigma_t}$$

one obtains

$$B^2 = 3\Sigma_t(\nu\Sigma_f - \Sigma_a). \quad (54)$$

Next we assume that the existence of a neutrons reflected and we introduce the justified approximation, (at least for U^{235} as can be seen from Eqs. (38) and (39)) that

$$\Sigma_f \approx \Sigma_a \quad \text{and} \quad \Sigma_c \approx 0 \quad (55)$$

this way we find that

$$B^2 = 3\Sigma_t\Sigma_f(\nu - 1). \quad (56)$$

The improved expression of B^2 based on the improved diffusion theory is obtained by a multiplication by a tamper factor Λ_0 given by

$$\Lambda_0 = \left[2 \left(1 + 0.3(\nu - 1) \frac{\sigma_f}{\sigma_t} \right) \right]^2. \quad (57)$$

Thus

$$B^2 = 3\Sigma_t\Sigma_f(\nu - 1)\Lambda_0 = 12\Sigma_t\Sigma_f(\nu - 1) \left(1 + 0.3(\nu - 1) \frac{\sigma_f}{\sigma_t} \right)^2. \quad (58)$$

Evaluating for the same values used earlier for U^{235} one finds that

$$\tilde{B}^2 = 3138.946(10)^{-4}. \quad (59)$$

For spherical geometry we leave

$$\tilde{B}^2 = \frac{\pi^2}{R^2} = 0.3138046$$

and thus

$$R = 5.608157 \text{ cm}. \quad (60)$$

Consequently for $\rho(U^{235}) = 19.05 \text{ g/cm}^3$ one finds the critical mass to be

$$M_c = \rho(4/3)\pi(5.608157)^3 = 14.07486 \text{ kg}. \quad (61)$$

This is very close to the most frequently quoted value in the literature namely $M_c \approx 15 \text{ kg}$

⁴Recent calculations shows that if U^{235} , P^{239} or CF^{249} is compressed to twice the normal density, then the critical mass is reduced to $M_c = 4.7 \text{ kg}$. $M_c = 1.8 \text{ kg}$ $M_c = 0.5 \text{ kg}$, respectively. Note also that $M_c(d_c^{10})^3 = M_c(2/3)^3 = (15)(2/3)^3 \approx 4.5 \text{ kg}$ is quite close to $M_c = 4.7 \text{ kg}$ of U^{235} with $\rho = 2(19.5) \text{ kg/cm}^3$. Our estimate $M_c = 14.07$ is very close to that of J. K. King.

for U^{235} . In praxis one would usually work with a sub-critical mass say 10 kg then by compressing it to a much higher density by reducing its radius from 5 to 4.3 cm for which ρ become equal to 28.6 g/cm^3 criticality is achieved for $M = 10 \text{ kg} < M_c = 15 \text{ kg}$. This is the implosion method.

It may be interesting to mention that Oppenheimer in [7] seems to have used a wrong value for ρ namely $\rho = 15$ and obtained the larger wrong value $M_c = 25 \text{ kg}$.

In other words if we use here $\rho = 15$ and change the values of the macroscopic cross-section accordingly we would have found $M_c = 25.16 \text{ kg}$ in accordance with Oppenheimer's original (inaccurate) analysis which over estimated M_c by more than 60%. In other words the critical mass depends sensitively on minor changes in the values of the physical parameters. That means finding the accurate material values experimentally is far more important than improving the differential equation of neutrons diffusion. We conclude by mentioning that in a forthcoming paper [12] we will be showing how all the preceding results may be obtained in a much simpler way and how they could be also improved using the theory of Cantorian spaces as well as the fractal space time approach to quantum physics [8-11].

II. Part Two - Self-weight

6. Self-weight buckling of an elastic column

The design of a long column with a high slenderness ratio [1] is in many ways similar to the problem of bringing two subcritical masses of a fissile material together to form one critical mass [11]. In both cases we are dealing with a critical state which may become fatally unstable [1,11].

The differential equation governing the buckling instability of an elastic column under its own weight is given by

$$EI\omega'' = p(l-x)\omega'; \quad (\cdot)' = d(\cdot)/dx, \quad (62)$$

where EI is the bending stiffness of the column, p the weight per unit length, l the height of the column, ω the lateral deflection, and x is the axial coordinate with $x = 0$ at the fixed bottom and $x=l$ at the free end of the column.

Eq. 62 is easily transferred to the Bessel differential equation

$$\mu'' + \frac{1}{Z}\mu' + \left(1 - \frac{1}{9Z^2}\right)\mu = 0, \quad (63)$$

where

$$\mu = \omega' \quad \text{and} \quad Z = \frac{2}{3} \sqrt{\frac{p}{EI}} (l-x)^3. \quad (64)$$

The solution of (63) can be expressed in terms of the Bessel functions and the critical buckling weight is easily found as an eigenvalue of (63) to be

$$\Lambda_i = (pl/EI) = (7.834401)/l^2. \quad (65)$$

We will come back shortly to this eigenvalue.

7. Fission critical mass for cylindrical geometry

The critical eigenvalue for fission in the case of a cylindrical geometrical configuration is easily obtained from a corresponding stationary Helmholtz equation as discussed above.

For the correct boundary conditions this is found just as in the self weight buckling problem using Bessel functions to be

$$B_g^2 = \left(\frac{\pi}{H}\right)^2 + \left(\frac{j_0}{R}\right)^2 = \left(\frac{\pi}{H}\right)^2 + \left(\frac{2.40483}{R}\right)^2. \quad (66)$$

For the optimal cylindrical shaped critical mass we have

$$H = R \left(\frac{\pi\sqrt{2}}{2.40483} \right) = R(1.84735). \quad (67)$$

Inserting in B_g^2 one finds

$$B_g^2 = \left[\frac{(\pi)(2.40483)}{(\pi)(\sqrt{2})(R)} \right]^2 + \left(\frac{2.40483}{R} \right)^2 = 16.8535/R^2. \quad (68)$$

The analogy is thus as follows:

	Self weight buckling	Self weight buckling
Physical	Ip / EI	$(\Sigma_f - \Sigma_a) / D$
Geometrical	l^2	R^2
Numerical constant	$a = 7.834401$	$b = 16.853501 = a(2.1512)$

8. The critical mass for cylindrical geometry

We have shown that the optimal or smallest critical mass for a cylindrical configuration is given by

$$H = R(1.84735). \quad (69)$$

We have also shown that efficiency of such a design is about 14% less than the spherical form. It is also not difficult to show that the implosion method is extremely simple for a cylindrical form. The explosive lensing design is in this case practically a one-dimensional problem. This is in contrast to the three-dimensional problem associated with spherical shapes. Therefore the explosive lensing is far simpler for a cylinder compared to a sphere. In Fig.1 we are showing schematically one such design. Such a design was most probably the basis for the first South African design which they used after the successful zero yield testing of the 55 kg 80% U^{235} "Melba" gun design at Avenda. This was between September 1979 and the beginning of 1980. President De Klerk disclosed the well-known "secret" at a joint session of parliament on 24 March 1993.

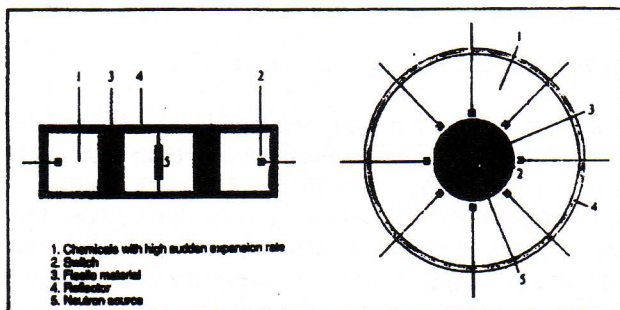


Fig. 1. A design for a fast fission device. On the left we have a side view of the optimal cylinder shown in a compressed "solid" state on the right. The critical mass is only 14% larger than that for a spherical geometry. For Pu^{239} we have $M_c = 5.7$ kg assuming normal density $\rho \approx 19.8 \text{ g/cm}^3$ and a reflector.

It may be mentioned at this point that South Africa seems to be the only country in the world who ever developed and then voluntarily gave up nuclear weapons. It is even claimed that they destroyed all their capabilities to design or fabricate such weapons. If this claim is true, then it must be an extraordinary example which every country in the world should follow.

There are some "historical" information that the final South African "gadget" was a 65 cm in diameter and 1.8 m in length. The weight is thought to have been about 1000 kg and used 55 kg of 90% U^{235} with an estimated yield of 10-18 KT. This is very low efficiency of about 1 to 1.8% which indicates a very conservative and "safe" design.

In conclusion we may just give an idea about the amount of compression needed to increase the density in order to achieve criticality.

To illustrate the idea let us regard a critical spherical mass of a fissile material of density $\rho = 18.5 \text{ g/cm}^3$ for which $M_c = S V_c = 50$ kg. Then the radius must be $R_c = 8.6411$ cm and volume is $V_c = 2702.7 \text{ cm}^3$. Suppose now we make from the material another sphere with a mass $M = 40$ kg. Such a mass is clearly subcritical. The subcritical radius of this mass is $R_1 = 8.02170$ cm and the volume is $V_1 = 2162.162162 \text{ cm}^3$.

Now the density which would make such a mass just as critical as $M_c = 50$ kg is found from

$$\rho_x(V_1) = (50)(1000). \quad (70)$$

That means

$$\rho_x = \frac{(50)(1000)}{2162.162162} = 23.125 \text{ g/cm}^3 \quad (71)$$

and the critical value is then

$$V_x = \frac{(18.5)(2162.16216)}{23.125} = (\rho_0 V_1) / S_x = 1729.7292 \text{ cm}^3. \quad (72)$$

Thus the critical radius which makes $m = 40$ kg critical is

$$R_x = \sqrt[3]{\frac{V_x}{(4/3)(\pi)}} = \sqrt[3]{\frac{(3)(1729.7292)}{4\pi}} = 7.446 \text{ cm}. \quad (73)$$

The radius reduction required is thus

$$\Delta R = 8.0217024 - 7.4466883 = R_i - R_x = 0.57501 \text{ cm.} \quad (74)$$

This is a reduction of about 7% which is easily achieved by conventional chemical explosives. In case of using U^{235} for which $M_c=15$ kg with tamper and, $\rho=19.05$ kg/cm³ one could make a 10 kg mass of U^{235} critical by reducing the radius from 5.728433 to 4.37716 cm that is to say a reduction of 12%. For a plutonium Pu^{239} the critical mass for ordinary density $\rho=19.84$ kg/cm³ is $M_c=15$ kg. A reduction, by 11.2% of the radius using implosion will reduce the critical mass to a 3.5 kg of Pu^{239} . For the sake of having an instructive comparison we give in Table 1 the critical mass for spherical geometry for various pure fissile materials.

Table 1

The theoretical critical mass for different fissile material in pure state (all the values are without reflector)

Fissile material	Critical mass (kg)
Pu238	9
Pu239	10.5
Pu240	41
Pu241	13
Pu242	88
Am241	115

Conclusions

There is an instructive analogy between the eigenvalue problems of structural engineering and applied mechanics and that of neutrons diffusion technology. In the present work we have given simple derivations to some previous classical results and corrected various errors which exist in the literature on the magnitude of the critical mass for fission. The knowledge of the exact value of the critical mass is of course extremely important for safe application of the corresponding technology in many engineering fields such as canals excavation, natural gas stimulation, mining and oil shale treatment [13].

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