

## BUCKLING AND INITIAL POST-BUCKLING BEHAVIOUR OF THIN-WALLED ELLIPTIC SHELLS UNDER BENDING

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The problem of the buckling and the initial post-buckling equilibrium paths of thin-walled cylindrical and elliptic shells subjected to bending has been carried out. Shell elements can be made of multi-layer orthotropic materials. The problem has been solved within the first order approximation of Koiter's asymptotic theory, using the transition matrix method.

*Keywords:* buckling, thin walled shells, bending.

### 1. Introduction

Vessels of tank trucks, tank wagons, as well as vessels of truck-wagons (bimodal tanks) can be examples of thin-walled structures built of thin cylindrical or elliptic shells [6]. A tendency towards a small mass of these structures causes that very thin shells of these vessels are exposed to a local loss of stability. Both freestanding horizontal vessels supported at their ends, as well as tank wagon vessels with a completely self-supporting structure are subject to bending and compression, it means, to loads that result in a loss of stability of the vessel shell. Particularly in the latest designs of tank wagons in which a vessel is fixed permanently with two skid rails (longitudinal thin-walled beams), the longitudinal forces are carried both by these skid rails and by the vessel shell. At considerable lengths of tank wagon vessels ( $L < 15$  m) and their large diameters ( $D < 3.2$  m), thin shells of these vessels are very supple to bending, and thus pliant to a loss of stability. Many aggressive chemicals require that the vessel shells should be made of composite materials. In the present paper an analysis of stability and initial post-buckling behaviour of thin-walled cylindrical and elliptic shells made of orthotropic materials subjected to bending and compression along the generating lines will be carried out.

An evaluation of the load-carrying capacity of the shells under analysis on the basis of the solutions obtained for the first and the second order non-linear approximation of Koiter's asymptotic theory of stability is planned as the next stage of these investigations.

## 2. Formulation of the problem

Let us consider cylindrical shell elements with closed or open cross-sections and with one axis of symmetry at least.

The shell elements under consideration can be multi-layer walls made of orthotropic materials, but the properties of the adjacent layer materials (the moduli of elasticity in particular) cannot differ in a radical way. The materials the shell elements are made of are subject to Hooke's law.

The shells under consideration can be loaded with bending in the plane of symmetry of the cross-section (flat bending). The aim of this study is to generate the stability equations and to solve the stability problem (to find the values of the critical load of local and global buckling), as well as to determine the initial equilibrium paths in the elastic post-buckling state for the shells.

## 3. Solution of the problem

The problem has been solved by the variational method using Koiter's asymptotic theory of conservative systems [3]. In the solution of the problem and in the computer program developed, the following have been employed: Byskov-Hutchinson's asymptotic expansion [1], the numerical transition matrix method [10] using Godunov's orthogonalization method [4-6].

The non-linear geometrical relationships for the cylindrical shell segment have been assumed in the form:

$$\begin{aligned}
 \varepsilon_1 &= u_{1,1} + 0.5u_{m,1}u_{m,1} \\
 \varepsilon_2 &= u_{2,2} + 0.5u_{m,2}u_{m,2} - k_2u_3 \\
 \varepsilon_3 &= 0.5(u_{1,2} + u_{2,1}) + 0.5u_{m,1}u_{m,2} \\
 \kappa_1 &= -u_{3,11} \\
 \kappa_2 &= -u_{3,22} - k_2u_{2,2} \\
 \kappa_3 &= 2u_{3,12} - k_2u_{2,1}
 \end{aligned} \tag{1}$$

where  $k_2 = 1/R = 2/D$  is the curvature of the cylindrical shell segment, and the summation with respect to the factor  $m$  is from 1 to 3 ( $m = 1, 2, 3$ ). The physical equations have been taken in the classical form as for linear-elastic orthotropic materials. It has been assumed that the hypothesis of the normal straight line (Kirchhoff's hypothesis) holds in multi-layer walls.

The equilibrium equations resulting from the virtual work principle have been obtained in the following form:

$$\begin{aligned}
 &[N_1(1+u_{1,1}) + N_3u_{1,2}]_{,1} + [N_2u_{1,2} + N_3(1+u_{1,1})]_{,2} = 0 \\
 &[N_1u_{2,1} + N_3(1+u_{2,2}) - k_2tN_6]_{,1} + [N_2(1+u_{2,2}) + N_3u_{2,1} - k_2tN_5]_{,1} = 0 \\
 &(tN_{4,1} + N_1u_{3,1} + N_3u_{3,2})_{,1} + (tN_{5,2} + 2tN_{6,1} + N_2u_{3,2} + N_3u_{3,1})_{,2} + k_2N_2 = 0
 \end{aligned} \tag{2}$$

In the above equations,  $N_1, N_2, N_3$  are the dimensionless sectional forces;  $N_4, N_5, N_6$  - the dimensionless sectional moments, whereas  $u_1 \equiv x, u_2 \equiv y, u_3 \equiv z$  - the components of the displacement vector in the  $x_1 \equiv x, x_2 \equiv y, x_3 \equiv z$  axis direction, respectively. The solution of these equations for each element should satisfy kinematic and static continuity conditions at the junctions of adjacent elements and the boundary conditions referring to the free support of the structure at its both ends, i.e.  $x=0$  and  $x=L$ .



As has been mentioned above, after expanding the fields of displacements  $\bar{U}_k$  and the fields of sectional forces  $\bar{N}_k$  into power series with respect to the buckling mode amplitudes  $\zeta_n$  (the amplitude of the  $n$ -th buckling mode divided by the thickness  $t$  of the wall assumed to be the first one), Koiter's asymptotic theory has been employed:

$$\begin{aligned}\bar{U}_k &= \lambda \bar{U}_k^{(0)} + \zeta_n \bar{U}_k^{(n)} + \dots \\ \bar{N}_k &= \lambda \bar{N}_k^{(0)} + \zeta_n \bar{N}_k^{(n)} + \dots\end{aligned}\quad (3)$$

where  $\bar{U}_k^{(0)}$ ,  $\bar{N}_k^{(0)}$  are the pre-buckling state fields, and  $\bar{U}_k^{(n)}$ ,  $\bar{N}_k^{(n)}$  – the  $n$ -th buckling mode fields. The range of indices is  $[1, J]$ , where  $J$  is the number of interacting modes.

After substitution of expansions (3) into the equilibrium equations (2), the junction conditions and the boundary conditions (corresponding to the free support at the vessel ends), the boundary problem of the zero (prebuckling state) and the first order approximation can be obtained.

The plates with linearly varying prebuckling stresses along their widths are divided into several strips under uniformly distributed compressive (tensile) stresses. Instead of the finite strip method, the exact transition matrix method is used in this case.

The zero approximation describes the pre-buckling state, whereas the first order approximation, being the linear problem of stability, allows for determination of values of critical loads, buckling modes, and initial post-buckling equilibrium paths [4-6]. The obtained system of homogeneous ordinary differential equations, with the corresponding conditions of the co-operation of elements, has been solved by the transition matrix method, having integrated numerically the equilibrium equations along the circumferential direction in order to obtain the relationships between the state vectors on two longitudinal edges. During the integration of the equations, Godunov's orthogonalization method is employed. The global buckling occurs at one sinusoid half-wave on the column length, whereas the local buckling takes place at the number of half-waves  $m > 1$ .

The developed computer program allows for a division of each shell element into several or even more than 40 strips made of different materials and with various thickness. A detailed description of the solution method of the problem under discussion, analogous as in the case of plate structures, has been included in Refs. [4-6].

At the point where the load parameter  $\lambda$  reaches its maximum value  $\lambda_s$  (secondary bifurcation or limit point) for the imperfect structure with regard to the imperfection of the buckling mode with the amplitude  $\zeta_r^*$ , the Jacobian of the non-linear system of equations [4,5]:

$$a_r (1 - \lambda / \lambda_r) \zeta_r + a_{jkr} \zeta_j \zeta_k + \dots = a_r \lambda / \lambda_r \zeta_r^* \quad \text{at } r = 1, 2, \dots, J \quad (4)$$

is equal to zero.

The index  $r$  is: 1 – for the global bulking mode;  $2 \dots J$  – for the local buckling modes.

The corresponding expression for the total elastic potential energy of the structures has the following form:

$$\Pi = -a_0 \lambda^2 / 2 + a_r (1 - \lambda / \lambda_r) \zeta_r^2 / 2 + a_{jkr} \zeta_j \zeta_k \zeta_r / 3 - a_r \zeta_r \zeta_r^* \lambda / \lambda_r \quad (5)$$

where:  $\lambda$  – load parameter,  $\lambda_r$  critical value of  $\lambda$ ,  $a_0 \lambda^2 / 2$  – energy of the pre-buckling state. Expressions for  $a_0$ ,  $a_r$ ,  $a_{jkr}$  are calculated by known formulae [4,5]. The formulae for the post-buckling coefficients  $a_{jkr}$  depend only on the buckling modes. Consideration of displacements and load components in the middle surface of the walls within the first order approximation, as well as more precise geometrical relationships enabled an analy-

sis of the shear-lag phenomenon, the distortions of cross-sections and all possible buckling modes, including a mixed buckling mode.

In the presented method, it is postulated that the reduced local critical load  $\lambda_r$  should be determined taking into account the global pre-critical bending  $\zeta_r^*=0$  within the first order non-linear approximation to the theory of the interactive buckling of the structure [1,8]. In order to find the lower bound estimation of the load-carrying capacity of thin-walled structures, the following assumptions have been made [8]:

- an interaction of only two modes of the global and local buckling within the first order approximation has been taken into account, i.e.  $J=2$ ;
- local imperfections are absent, i.e.  $\zeta_2^*=0$ .

If we take into account the above-mentioned assumptions, Eq. (4) leads to the following set of algebraic equations of equilibrium [8]:

$$\begin{aligned} a_1(1-\lambda/\lambda_1)\zeta_1 + a_{122}\zeta_2^2 &= a_1\zeta_1^*\lambda/\lambda_1 \\ a_2(1-\lambda/\lambda_2)\zeta_2 + 2a_{122}\zeta_1\zeta_2 &= 0 \end{aligned} \quad (6)$$

All coefficients with  $j, k, r > 1$  are equal to zero and non-zero coefficients are only those that have one index equal to 1 and an even sum of  $(j+k)$ . If we introduce the following notation:

$$\vartheta = (\lambda/\lambda_2 - 1)/\zeta_1 \quad (7)$$

the second of equations (6) can be written in the form of an eigenvalue problem:

$$(2a_{122}/a_2 - \vartheta)\zeta_2 = 0 \quad (8)$$

In the pre-buckling state, the single solution to Eq. (8) is a trivial solution  $\zeta_2=0$  and only the overall deflection develops according to the first of Eq. (6):

$$\zeta_1 = \zeta_1^*\lambda/(\lambda_1 - \lambda) \quad (9)$$

The coupled (interactive) buckling with simultaneous overall and local deflections becomes possible when there appears a non-zero solution  $\zeta_2 \neq 0$ , to the set of Eqs. (6). Note that the sign of  $\vartheta=2a_{122}/a_2$  determines the direction of the overall deflection according to the condition  $\vartheta\zeta_1 < 0$ .

The eigenvector from Eq. (8) has been determined with accuracy up to the constant  $C$  and it has been normalised with the condition  $\left[ \left( \zeta_2^o \right)^2 \right]^{1/2} = 1$ .

As the initial post-buckling path for the first order approximation always falls, the maximal value  $\lambda=\lambda_R$  corresponds to the value  $C=0$  (the point of intersection of the pre-buckling path (9) and the initial post-buckling path (7)). Then:

$$\left[ \zeta_1^* \frac{\lambda_R}{\lambda_1} + \left( 1 - \frac{\lambda_R}{\lambda_1} \right) \left( 1 - \frac{\lambda_R}{\lambda_2} \right) \frac{1}{\vartheta} \right] = 0 \quad (10)$$

The maximum load value  $\lambda_R$  determined from Eq. (10) is smaller than the critical value of the local buckling  $\lambda_2$ . The load  $\lambda_R$  can be interpreted as an influence of the load corresponding to the global buckling ( $\zeta_1^* \neq 0$ ,  $\zeta_1 \neq 0$ ) on the critical value of the local load  $\zeta_2 \neq 0$ . Thus, the critical load corresponding to  $\lambda_R$  can be called *the reduced critical load value* of the local buckling.

#### 4. Analysis of the problem

A detailed analysis has been carried out for elliptic vessels subjected to pure bending. The following geometrical dimensions of the vessels have been assumed:



- cylindrical vessel [7]:  $k=b/a=1$   
 $R=D/2=a=b=1300.0$  mm,  $h=3$  mm;
- elliptic vessel:  $k=b/a=0.8$   
 $a=1435.6$  mm,  $b=1148.5$  mm,  $h=3$  mm;
- elliptic vessel:  $k=b/a=6$   
 $a=1576.5$  mm,  $b=945.9$  mm,  $h=3$  mm;

where  $a$  and  $b$  are the mean lengths of the ellipsis axes, whereas  $k=b/a$  is the affinity coefficient of the ellipsis with respect to the semi-major axis. It has been assumed that the vessels made of an isotropic material with the following material constants:

$$G = \frac{E}{2(1+\nu)}; \quad \nu = 0.3,$$

are subject to pure bending with respect to the semi-major axis of the ellipsis.

In Fig. 1 the dimensionless values of the critical load  $\sigma^* = 10^3/E$  for the vessels under analysis as a function of the number of half-waves  $m$  that generate along the vessel length when the vessel length is assumed to be  $L=12500$  mm (identically as in [7]) are shown. The critical values  $\sigma^*$  corresponding to the number of half-waves  $m=1$  are the critical values of the global buckling of the vessels under bending, whereas for  $m>1$  they are the critical values of the local buckling. The assumed theory and the problem solution method allows one to describe the effect occurring during the global buckling of shells under bending, i.e. the Brazier effect [9].

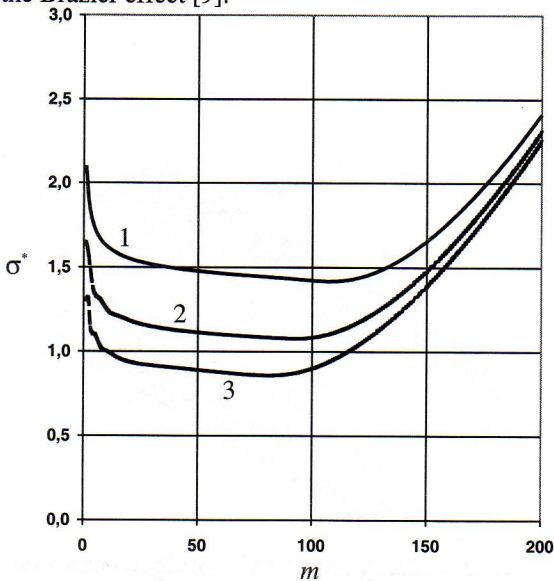


Figure 1 Dimensionless critical stress  $\sigma^*$  carried by the number of half-waves  $m$ . Curve 1:  $k=1.0$ ; 2:  $k=0.8$ ; 3:  $k=0.6$

In Table 1 the obtained values of critical loads and the respective values of bending moments on the assumption that  $E=200$  GPa are given. Additionally, the number of half-waves  $m$  that generate along the vessel length for the values of the local critical load is given as well.

Table 1. Critical values of loads for the vessels under bending

	Global buckling		Local buckling	
	$\sigma_g^* \equiv \sigma_g^*$	$M$ [kNm]	$\sigma_1^* \equiv \sigma_1^*$	$M$ [kNm]
$k = 1.0$	2.106	6700	1.405 ( $m=113$ )	4470
$k = 0.8$	1.853	5460	1.064 ( $m=98$ )	3135
$k = 0.6$	1.538	4535	0.847 ( $m=88$ )	2495

To verify the presented values of critical loads, the numerical calculations were made with the well-known commercial *ANSYS v.5.7* Finite Elements package. In order to simulate a loading condition that gives origin to bending moment only, a stiffened ring was applied at both ends of the tube, identically as was done by Guarracino [2]. A good agreement of local values of critical loads obtained by means of these two methods was reached. The computational results referring to the cylindrical shell for  $L/D=2.0$  are presented below. The critical load value obtained with MES is equal to  $M_{cr}=4873$  [kNm], its respective buckling mode is shown in Fig. 2, whereas the theoretical analysis result is  $M_{cr}=4470$  [kNm] (see Table 1 – case  $k=1.0$ ).

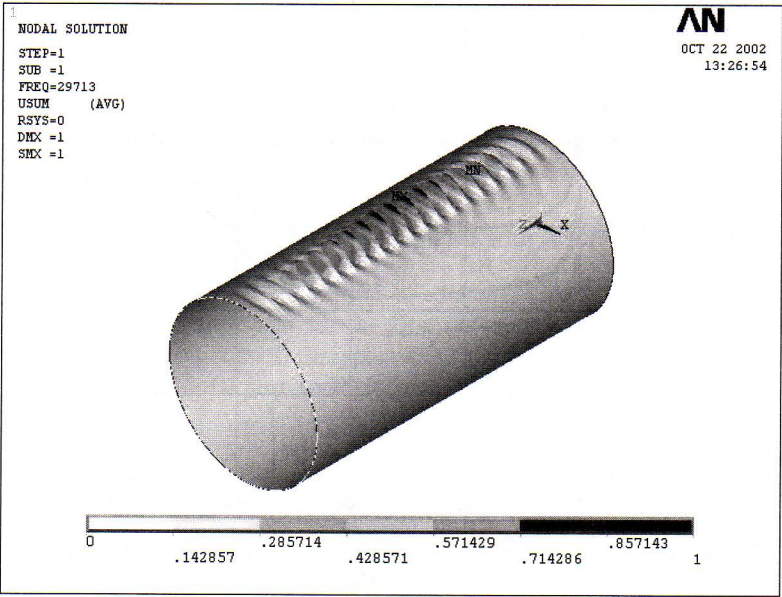


Figure 2 Buckling mode by ANSYS5.7

The subsequent three figures (Fig. 3–5) show: the ratio of the global critical load to the lowest value of the local critical load  $\sigma_g^* / \sigma_1^*$  (curve 1), the ratio of the reduced local critical load to the lowest local critical load  $\sigma_R^* / \sigma_1^*$  for two values of the global imperfection  $\sigma_g^* = L/(1000h)$  (curve 2),  $\sigma_g^* = 1.0$  (curve 3), and the ratio of the theoretical load-carrying capacity within the first order non-linear approximation to the lowest local

critical load  $\sigma_s^*/\sigma_1^*$  for the assumed imperfections:  $\zeta_g^* = \left| \frac{L}{1000h} \right|$ ,  $\xi_1^* = |0.2|$ , as a function of the vessel length  $L/D=L/(2R)$  (curve 4) for  $k=1.0$ , 0.8 and 0.6, respectively.

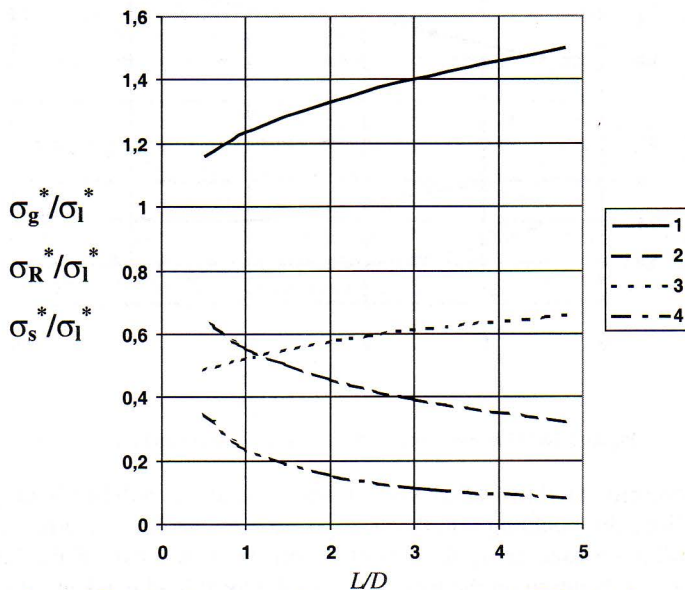


Figure 3 Relationship between dimensionless stresses and  $L/D$  for  $k=1.0$

It should be emphasised that the lowest values of the local critical loads  $\sigma_1^*$  are constant in practice for the length variability range under consideration ( $0.5 \leq L/D < 4.8$ ). The assumed level of the global imperfections  $\sigma_g^* = L/(1000h)$  is the most often assumed value of imperfections of thin-walled structures in Eurocodes.

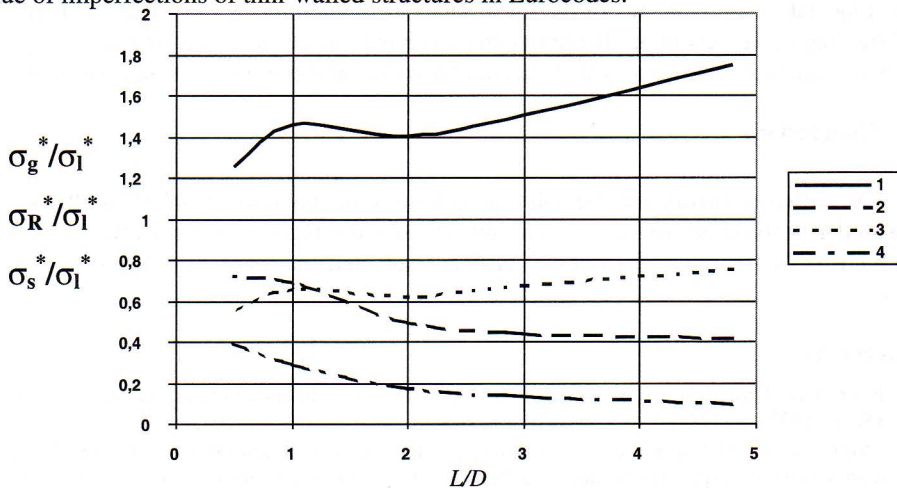


Figure 4 Relationship between dimensionless stresses and  $L/D$  for  $k=0.8$



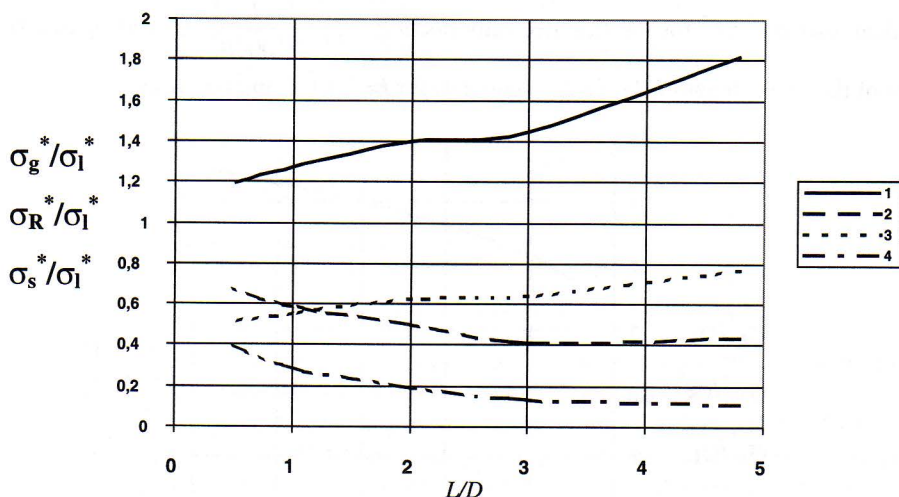


Figure 5 Relationship between dimensionless stresses and  $L/D$  for  $k=0.6$

In the majority of thin-walled vessels under bending, bulkheads or partitions that most often limit the buckling length corresponding to the Brazier effect up to approximately  $L/D=0.5$  are used along the vessel length. In such a case if the influence of the global pre-critical bending on the local value of the local load is taken into consideration, it reduces the critical load up to 30-50% for the magnitudes of imperfections under investigation, whereas in the case the coupled buckling is taken into consideration - up to 60%. It should be remembered that the determined values of the local critical loads refer to the upper critical load. Thus, if the global pre-critical bending or the coupled bending is taken into account, then it is possible to estimate better the load-carrying capacity in the post-buckling state than by means of the linear analysis of stability. A more thorough analysis of buckling of vessels under bending can be carried out when the second order approximation is taken into account, which has not been the subject of the present investigations.

## 5. Conclusions

The presented theory and the solution method of the buckling problem of elliptic vessels under bending allows one to take into account the Brazier effect, as well as to estimate correctly the reduced values of critical loads and the theoretical load-carrying capacity.

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