

ON A CUSPED ELASTIC SOLID-INCOMPRESSIBLE FLUID INTERACTION PROBLEM. HARMONIC VIBRATION

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Admissible static and dynamical problems are investigated for a cusped plate. The setting of boundary conditions at the plates ends depends on the geometry of sharpenings of plates ends, while the setting of initial conditions is independent of them. Interaction problem between an elastic cusped plate and viscous incompressible fluid is studied.

Keywords: solid-fluid interaction, cusped plate, degenerate ordinary differential equation, degenerate hyperbolic equation, boundary value problems, vibration.

1. Introduction

In 1955 I.Vekua [22]– [24] raised the problem of investigation of cusped plates, i.e. such ones whose thickness on the part of plate boundary or on the whole one vanishes. The problem mathematically leads to the question of setting and solving of boundary value problems (BVP) for even order equations and systems of elliptic type with the order degeneration in the statical case and of initial boundary value problems (IBVP) for even order equations and systems of hyperbolic type with the order degeneration in the dynamical case. There exists a wide literature devoted to the theory of degenerate and mixed type equations (see, e.g., [2, 7, 18]), which was developed intensively in the period from early 50-ies till early 70-ies but it could not cover the above equations and systems because of distinct peculiarities of the latters caused by the geometry of the mechanical problem.

The first work concerning classical bending of cusped elastic plates was done by S. Mikhlin [16] and Makhover [14, 15]. Since, a lot of sciences were investigated cusped plates and shells, see, e.g., [4, 5, 9, 10]. A brief survey of results and references can be found in [11]. In the last period various authors dedicated their work to solid-fluid interaction problems and to applications of such problems to the engineering.

The aim of the present paper is to consider interaction between an elastic cusped plate and fluid. We will consider a plate, whose projection on $x_3 = 0$ occupies the

domain Ω

$$\Omega = \{(x_1, x_2, x_3) : -\infty < x_1 < \infty, 0 < x_2 < l, x_3 = 0\}.$$

The equation of bending vibration has the following form (see, e.g., [20])

$$(D(x_2)w,_{22}(x_2, t)),_{22} = q(x_2, t) - 2\rho h(x_2)\frac{\partial^2 w(x_2, t)}{\partial t^2}, \quad 0 < x_2 < l, \quad (1)$$

where $w(x_2)$ is a deflection of the plate, $q(x_2)$ is an intensity of a lateral load, ρ is a density of the shell, $D(x_2)$ is a flexural rigidity,

$$D(x_2) := \frac{2Eh^3(x_2)}{3(1-\nu^2)}, \quad (2)$$

where E is the Young's modules, ν is the Poisson's ratio, and $2h(x_2)$ is the thickness of the shell. Let $E = \text{const}$, $\nu = \text{const}$, and

$$D(x_2) = D_0 x_2^\alpha (l - x_2)^\beta, \quad D_0, \alpha, \beta = \text{const}, \quad D_0 > 0, \quad \alpha, \beta \geq 0. \quad (3)$$

Then

$$2h(x_2) = h_0 x_2^{\alpha/3} (l - x_2)^{\beta/3}, \quad h_0 = \text{const} > 0.$$

In the case $\alpha^2 + \beta^2 > 0$ equation (1) becomes degenerate one. Such plates are called cusped plates.

In the case under consideration (see [20])

$$M_2(x_2, t) := -D(x_2)w,_{22}(x_2, t), \quad (4)$$

$$Q_2(x_2, t) := M_{2,2}(x_2, t), \quad (5)$$

where $M_2(x_2, t)$ is a bending moment, $Q_2(x_2, t)$ is an intersecting force.

Section 2 is devoted to the investigation of properties of solution of the equation (1) and formulation of all admissible BVPs for harmonic vibration of cusped elastic plates.

Section 3 deals with incompressible fluid – elastic plates interactions problems.

2. Harmonic Vibration of an Elastic Cusped Plates

In case of cylindrical bending (1), (4) and (5) becomes

$$(D(x_2)w,_{22}(x_2)),_{22} = q(x_2), \quad 0 < x_2 < l, \quad (6)$$

$$M_2(x_2) = -D(x_2)w,_{22}, \quad Q_2(x_2) = M_{2,2}(x_2). \quad (7)$$

Obviously, if we suppose $q(x_2) \in C([0, l])$ we have

$$Q_2(x_2) := - \int_{x_2^0}^{x_2} q(\xi) d\xi - c_1, \quad (8)$$

$$M_2(x_2) := - \int_{x_2^0}^{x_2} (x_2 - \xi) q(\xi) d\xi - c_1 x_2 - c_2, \quad (9)$$

$$w_{,2}(x_2) : = \int_{x_2^0}^{x_2} \left\{ - \left[\int_{x_2^0}^{\xi} \eta q(\eta) d\eta + c_2 \right] + \xi \left[\int_{x_2^0}^{\xi} q(\eta) d\eta + c_1 \right] D^{-1}(\xi) \right\} d\xi + c_3, \quad (10)$$

$$w(x_2) := \int_{x_2^0}^{x_2} (x_2 - \xi) \left\{ - \left[\int_{x_2^0}^{\xi} \eta q(\eta) d\eta + c_2 \right] + \xi \left[\int_{x_2^0}^{\xi} q(\eta) d\eta + c_1 \right] D^{-1}(\xi) \right\} d\xi + c_3 x_2 + c_4, \quad x_2^0 \in]0, l[. \quad (11)$$

At points 0, l all above quantities are defined as the corresponding limits when $x_2 \rightarrow 0_+$ and $x_2 \rightarrow l_-$.

Remark 2.1 Since $q(x_2) \in C([0, l])$, according to (8)-(11), we obtain $w(\cdot, t) \in C^4([0, l])$, and

$$Q_2(\cdot, t), \quad M_2(\cdot, t) \in C([0, l]), \\ w(\cdot, t), \quad w_{,2}(\cdot, t) \in C([0, l]),$$

the behaviour of the $w_{,2}(x_2)$ and $w(x_2)$ when $x_2 \rightarrow 0_+$ and $x_2 \rightarrow l_-$ depends on α and β , as follows:

$$\begin{array}{ll} w \in C^1([0, l]) \quad (w \in C^1((0, l))) & \text{if } \alpha < 1, \beta > 1 \quad (\alpha > 1, \beta < 1); \\ w \in C([0, l]) \quad (w \in C((0, l))) & \text{if } \alpha < 2, \beta > 2 \quad (\alpha > 2, \beta < 2); \\ w \in C^1([0, l]) & \text{if } \alpha, \beta < 1; \\ w \in C([0, l]) & \text{if } \alpha, \beta < 2; \end{array}$$

$$w \in C^1([0, l]) \cap C([0, l]), \quad (w \in C^1((0, l]) \cap C([0, l])) \\ \text{if } \alpha < 1, \beta < 2 \quad (\alpha < 2, \beta < 1).$$

Taking into account of above remark, we have following problems

Problem 1 Let $0 \leq \alpha < 1$, $0 \leq \beta < 1$. Find $w \in C^4([0, l]) \cap C^1([0, l])$ satisfying (1) and the following boundary conditions (BCs):

$$w(0) = g_{11}, \quad w_{,2}(0) = g_{21}, \quad w(l) = g_{12}, \quad w_{,2}(l) = g_{22};$$

Problem 2 Let $0 \leq \alpha < 1$, $0 \leq \beta < 1$. Find $w \in C^4([0, l]) \cap C^1([0, l])$ satisfying (1) and BCs:

$$w(0) = g_{11}, \quad w_{,2}(0) = g_{21}, \quad w_{,2}(l) = g_{22}, \quad Q_2(l) = h_{22};$$

Problem 3 Let $0 \leq \alpha < 1$, $0 \leq \beta < 2$. Find $w \in C^4([0, l]) \cap C^1([0, l]) \cap C([0, l])$ satisfying (1) and BCs:

$$w(0) = g_{11}, \quad w_{,2}(0) = g_{21}, \quad w(l) = g_{12}, \quad M_2(l) = h_{12};$$

Problem 4 Let $0 \leq \alpha < 1$, $\beta \geq 0$. Find $w \in C^4([0, l]) \cap C^1([0, l])$ satisfying (1) and the following BCs:

$$w(0) = g_{11}, \quad w_{,2}(0) = g_{21}, \quad M_2(l) = h_{12}, \quad Q_2(l) = h_{22};$$

Problem 5 Let $0 \leq \alpha$, $\beta < 1$. Find $w \in C^4([0, l]) \cap C^1([0, l])$ satisfying (1) and the following BCs:

$$w_{,2}(0) = g_{21}, \quad Q_2(0) = h_{21}, \quad w(l) = g_{12}, \quad w_{,2}(l) = g_{22};$$

Problem 6 Let $0 \leq \alpha < 1$, $0 \leq \beta < 2$. Find $w \in C^4([0, l]) \cap C^1([0, l]) \cap C([0, l])$ satisfying (1) and the following BCs:

$$w_{,2}(0) = g_{21}, \quad Q_2(0) = h_{21}, \quad w(l) = g_{12}, \quad M_2(l) = h_{12};$$

Problem 7 Let $0 \leq \alpha < 2$, $0 \leq \beta < 1$. Find $w \in C^4([0, l]) \cap C^1([0, l]) \cap C([0, l])$ satisfying (1) and the following BCs:

$$w(0) = g_{11}, \quad M_2(0) = h_{11}, \quad w(l) = g_{12}, \quad w_{,2}(l) = g_{22};$$

Problem 8 Let $0 \leq \alpha < 2$, $0 \leq \beta < 1$. Find $w \in C^4([0, l]) \cap C([0, l]) \cap C^1([0, l])$ satisfying (1) and the following BCs:

$$w(0) = g_{11}, \quad M_2(0) = h_{11}, \quad w_{,2}(l) = g_{22}, \quad Q_2(l) = h_{22};$$

Problem 9 Let $0 \leq \alpha$, $\beta < 2$. Find $w \in C^4([0, l]) \cap C([0, l])$ satisfying (1) and the following BCs:

$$w(0) = g_{11}, \quad M_2(0) = h_{11}, \quad w(l) = g_{12}, \quad M_2(l) = h_{12};$$

Problem 10 Let $\alpha \geq 0$, $0 \leq \beta < 1$. Find $w \in C^4([0, l]) \cap C^1([0, l])$ satisfying (1) and the following BCs:

$$M_2(0) = h_{11}, \quad Q_2(0) = h_{22}, \quad w(l) = g_{12}, \quad w_{,2}(l) = g_{22}.$$

In all these problems $g_{i,j}$, $h_{i,j}$ ($i, j = 1, 2$) are given constants.

All the above problems are solved in the explicit forms.

Remark 2.2 Problems 1-10 are not correct for the different values of α and β indicated in Problems 1-10. It is evident from the fact that in the above cases, in general, the limits of w and $w_{,2}$ as $x_2 \rightarrow 0_+$, l_- do not exist. The last assertions easily follow from the general representations (11) of w and $w_{,2}$ with (10).

In case of homogeneous BCs solution of all the above problems can be represented as follows [6]

$$w(x_2) = \int_0^l K(x_2, \xi) q(\xi) d\xi, \quad (12)$$

where

$$K(x_2, \xi) = \begin{cases} K_3(\xi, x_2), & 0 \leq \xi \leq x_2, \\ K_3(x_2, \xi), & x_2 \leq \xi \leq l. \end{cases} \quad (13)$$

$K_3(x_2, \xi)$ has different forms for different problems, e.g.,

Problem 1.

$$\begin{aligned} K_3(x_2, \xi) = & \int_0^{x_2} (\eta - x_2)(\eta - \xi) D^{-1}(\eta) d\eta \\ & + \left\{ \int_0^{\xi} (\xi - \eta) D^{-1}(\eta) d\eta \int_0^{x_2} (x_2 - \eta) \eta D^{-1}(\eta) d\eta \right. \\ & + \int_0^{\xi} \eta (\xi - \eta) D^{-1}(\eta) d\eta \int_0^{x_2} (x_2 - \eta) D^{-1}(\eta) d\eta \left. \right\} \frac{\int_0^l \eta D^{-1}(\eta) d\eta}{\Delta} \\ & - \int_0^{\xi} (\xi - \eta) \eta D^{-1}(\eta) d\eta \int_0^{x_2} (x_2 - \eta) \eta D^{-1}(\eta) d\eta \frac{\int_0^l D^{-1}(\eta) d\eta}{\Delta} \\ & + \int_0^{\xi} (\xi - \eta) D^{-1}(\eta) d\eta \int_0^{x_2} (x_2 - \eta) D^{-1}(\eta) d\eta \frac{\int_0^l \eta^2 D^{-1}(\eta) d\eta}{\Delta}, \end{aligned} \quad (14)$$

where

$$\Delta := \left[\int_0^l \xi D^{-1}(\xi) d\xi \right]^2 - \int_0^l D^{-1}(\xi) d\xi \int_0^l \xi^2 D^{-1}(\xi) d\xi < 0.$$

The last assertion follows from the Hölder inequality which is strong since $\xi D^{-\frac{1}{2}}(\xi)$ and $D^{-\frac{1}{2}}(\xi)$ are positive on $]0, l[$, and $\xi^2 D^{-1}(\xi)$ and $D^{-1}(\xi)$ differ from each other by a nonconstant factor ξ^2 .

Problem 2.

$$\begin{aligned} K_3(x_2, \xi) = & \int_0^{x_2} (x_2 - \eta)(\xi - \eta) D^{-1}(\eta) d\eta \\ & - \frac{1}{\int_0^l (l - \eta) D^{-1}(\eta) d\eta} \int_0^{\xi} (\xi - \eta) D^{-1}(\eta) d\eta \\ & \times \int_0^{x_2} (x_2 - \eta) D^{-1}(\eta) d\eta. \end{aligned} \quad (15)$$

Problem 3.

$$\begin{aligned}
K_3(x_2, \xi) &= \int_0^{x_2} (x_2 - \eta)(\xi - \eta)D^{-1}(\eta)d\eta \\
&\quad - \frac{1}{\int_0^l (l - \eta)^2 D^{-1}(\eta)d\eta} \int_0^{x_2} (x_2 - \eta)(l - \eta)D^{-1}(\eta)d\eta \\
&\quad \times \int_0^{\xi} (\xi - \eta)(l - \eta)D^{-1}(\eta)d\eta.
\end{aligned} \tag{16}$$

Problem 4.

$$K_3(x_2, \xi) = \int_0^{x_2} (\xi - \eta)(x_2 - \eta)D^{-1}(\eta)d\eta. \tag{17}$$

Problem 5.

$$\begin{aligned}
K_3(x_2, \xi) &= \int_{x_2}^l (x_2 - \eta)(\xi - \eta)D^{-1}(\eta)d\eta \\
&\quad - \frac{1}{\int_0^l \eta D^{-1}(\eta)d\eta} \int_{x_2}^l (x_2 - \eta)D^{-1}(\eta)d\eta \\
&\quad \times \int_{\xi}^l (\xi - \eta)D^{-1}(\eta)d\eta.
\end{aligned} \tag{18}$$

Problem 6.

$$\begin{aligned}
K_3(x_2, \xi) &= -(l - x_2) \int_{\xi}^{x_2} \eta D^{-1}(\eta)d\eta + \int_l^{x_2} \eta^2 D^{-1}(\eta)d\eta \\
&\quad + (l - x_2)\xi \int_{\xi}^0 D^{-1}(\eta)d\eta.
\end{aligned} \tag{19}$$

Problem 7.

$$\begin{aligned}
K_3(x_2, \xi) &= \int_{\xi}^l (x_2 - \eta)(\xi - \eta)D^{-1}(\eta)d\eta \\
&\quad - \frac{1}{\int_0^l \eta^2 D^{-1}(\eta)d\eta} \int_{x_2}^l (x_2 - \eta)\eta D^{-1}(\eta)d\eta
\end{aligned}$$

$$\times \int_{\xi}^l (\xi - \eta) \eta D^{-1}(\eta) d\eta. \quad (20)$$

Problem 8.

$$\begin{aligned} K_3(x_2, \xi) := & -x_2 \int_{\xi}^{x_2} \eta D^{-1}(\eta) d\eta + \int_0^{x_2} \eta^2 D^{-1}(\eta) d\eta \\ & + x_2 \xi \int_{\xi}^l D^{-1}(\eta) d\eta. \end{aligned} \quad (21)$$

Problem 9.

$$\begin{aligned} K_3(x_2, \xi) = & \frac{x_2 \xi}{l^2} \int_{\xi}^l (l - \eta) D^{-1}(\eta) d\eta + \frac{x_2(l - \xi)}{l^2} \int_{\xi}^{x_2} (l - \eta) \eta D^{-1}(\eta) d\eta \\ & + \frac{(l - x_2)(l - \xi)}{l^2} \int_0^{x_2} \eta^2 D^{-1}(\eta) d\eta. \end{aligned} \quad (22)$$

Problem 10.

$$K_3(x_2, \xi) = - \int_{\xi}^l (x_2 - \eta)(\eta - \xi) D^{-1}(\eta) d\eta. \quad (23)$$

Obviously, taking into account (14)-(23), we have (see (13))

$$K(x_2, \xi) \in \begin{cases} C([0, l] \times [0, l]), & \text{in case of Problems 1 - 3, 5 - 9;} \\ C([0, l] \times]0, l]), & \text{in case of Problems 10;} \\ C([0, l[\times [0, l]), & \text{in case of Problems 4,} \end{cases} \quad (24)$$

and

$$K'_{,2}(x_2, \xi) \in \begin{cases} C([0, l] \times [0, l]), & \text{in case of Problems 1, 2, 5, 9;} \\ C([0, l] \times]0, l]), & \text{in case of Problems 7, 8, 10;} \\ C([0, l[\times [0, l]), & \text{in case of Problems 3, 4, 6;} \end{cases} \quad (25)$$

Let us consider equation (1) in the case of harmonic vibration. In this case

$$w(x_2, t) = e^{i\omega t} w_0(x_2), \quad q(x_2, t) = e^{i\omega t} q_0(x_2), \quad (26)$$

where $\omega = \text{const}$ is an oscillation frequency, $q_0(x_2) \in C([0, l])$ is a given function. Now, for $w_0(x_2)$ from (1), we get the following equation

$$(D(x_2)w_0''(x_2))'' = q_0(x_2) + 2\omega^2 \rho h(x_2)w_0(x_2), \quad (27)$$

which we solve under the above BVPs (see problems 1–10), where we replace $w(x_2)$ and $w'(x_2)$ by $w_0(x_2)$ and $w'_0(x_2)$ with

$$M_2(x_2) = -D(x_2)w_{0,22}, \quad Q_2(x_2) = M_{2,2}(x_2).$$

All these problems are equivalent to the following integral equation (which we get from (12) after replacing $w(x_2)$ and $q(x_2)$ by $w_0(x_2)$ and $q_0(x_2) + 2\omega^2 \rho h(x_2)w_0(x_2)$), respectively

$$w_0(x_2) - \omega^2 \int_0^l K(x_2, \xi) g(\xi) w_0(\xi) d\xi = F(x_2), \quad (28)$$

where $g(\xi) := 2\rho h(x_2)$

$$F(x_2) := \int_0^l K(x_2, \xi) q_0(\xi) d\xi.$$

Proposition 2.3 $K(x_2, \xi)$ is a symmetric with respect to x_2 and ξ .

Proof. For z_1 and z_2 , such that $0 \leq z_1, z_2 \leq l$ we have

$$K(z_1, z_2) = \begin{cases} K_3(z_2, z_1), & 0 \leq z_2 \leq z_1 \leq l, \\ K_3(z_1, z_2), & 0 \leq z_1 \leq z_2 \leq l, \end{cases}$$

$$K(z_2, z_1) = \begin{cases} K_3(z_1, z_2), & 0 \leq z_1 \leq z_2 \leq l, \\ K_3(z_2, z_1), & 0 \leq z_2 \leq z_1 \leq l, \end{cases}$$

i.e.,

$$K(z_1, z_2) = K(z_2, z_1), \quad \text{for any } z_1, z_2 \in [0, l].$$

◇

Introducing a new unknown function

$$w_1(x_2) = w_0(x_2) \sqrt{g(x_2)} \quad (29)$$

we can reduce (28) to the following integral equation

$$w_1(x_2) - \omega^2 \int_0^l R(x_2, \xi) w_1(\xi) d\xi = F(x_2) \sqrt{g(x_2)} \quad (30)$$

where

$$R(x_2, \xi) := \sqrt{g(x_2)} K(x_2, \xi) \sqrt{g(\xi)}. \quad (31)$$

Obviously, (30) is an integral equation with a symmetric and continuous kernel.

Let us denote by λ_n eigenvalues of $R(x_2, \xi)$ corresponding to eigenfunction Y_n , i.e.,

$$Y_n(x_2) = \lambda_n \int_0^l R(x_2, \xi) Y_n(\xi) d\xi, \quad (32)$$

and let

$$X_n := \frac{Y_n(x_2)}{\sqrt{g(x_2)}}. \quad (33)$$

Further, in view of (31) and (32) we have

$$X_n(x_2) = \lambda_n \int_0^l g(\xi) K(x_2, \xi) X_n(\xi) d\xi. \quad (34)$$

It is easy to prove the following propositions (see [6])

Proposition 2.4 $Y_n(x_2) \in C^4([0, l])$.

Proposition 2.5 Let $Y_n(x_2) \in C^4([0, l])$. Number of eigenvalues λ_n of (30) is not finite.

Let recall the following well-known theorem (see [13])

Theorem 2.6 If $u(x_2)$ has the form

$$u(x_2) = \lambda \int_0^l R(x_2, \xi) f(\xi) d\xi,$$

with $f(x_2)$ continuous on $[0, l]$, and a symmetric kernel $R(x_2, \xi) \in C([0, l] \times [0, l])$, then

$$u(x_2) = \sum_{n=1}^{\infty} (u, Y_n) Y_n(x_2), \quad (35)$$

where

$$(u, Y_n) := \int_0^l u(x_2) Y_n(x_2) dx_2,$$

Y_n is an eigenfunction of $R(x_2, \xi)$, and the series on the right hand side of (35) is convergent absolutely and uniformly on $[0, l]$.

Proposition 2.7 All λ_n are positive.

If $\omega^2 \neq \lambda_n$, the unique solution of (30) can be written as follows (see, e.g., [13], Theorem XVIII, p.157)

$$\begin{aligned} w_1(x_2) &= F(x_2) \sqrt{g(x_2)} \\ &+ \omega^2 \sum_{n=1}^{\infty} \left[\frac{1}{\lambda_n - \omega^2} \int_0^l F(\xi) \sqrt{g(\xi)} Y_n(\xi) d\xi \right] Y_n(x_2), \end{aligned} \quad (36)$$

where the series in the right hand side of (36) is absolutely and uniformly convergent on $[0, l]$.

After substituting (29) and (36), into (33) we formally have

$$w_0(x_2) = F(x_2) + \omega^2 \sum_{n=1}^{\infty} \left[\frac{1}{\lambda_n - \omega^2} \int_0^l F(\xi) \sqrt{g(\xi)} Y_n(\xi) d\xi \right] X_n(x_2). \quad (37)$$

We have to prove that (37) is a solution of (27) under BCs 1–10.

Let differentiate (37) formally i -times with respect to x_2 and consider the following expressions

$$w_0^{(i)}(x_2) = F^{(i)}(x_2) + \omega^2 \sum_{n=1}^{\infty} \left[\frac{1}{\lambda_n - \omega^2} \int_0^l F(\xi) \sqrt{g(\xi)} Y_n(\xi) d\xi \right] X_n^{(i)}(x_2), \quad (38)$$

$i = 1, \dots, 4$

Proposition 2.8 *The series on the right hand side of (37) and (38) are absolutely and uniformly convergent on $]0, l[$.*

Proof. Let denote by

$$S_n(x_2) := \frac{1}{\lambda_n - \omega^2} \int_0^l F(\xi) \sqrt{g(\xi)} Y_n(\xi) d\xi X_n(x_2).$$

Taking into account (34), (33) we have

$$\begin{aligned} S_n(x_2) &= \frac{\lambda_n}{\lambda_n - \omega^2} \int_0^l F(\xi) \sqrt{g(\xi)} Y_n(\xi) d\xi \int_0^l g(\eta) K(x_2, \eta) X_n(\eta) d\eta \\ &= \frac{\lambda_n}{\lambda_n - \omega^2} \int_0^l F(\xi) \sqrt{g(\xi)} Y_n(\xi) d\xi \int_0^l \sqrt{g(\eta)} K(x_2, \eta) Y_n(\eta) d\eta \\ &= \frac{\lambda_n}{\lambda_n - \omega^2} \int_0^l \sqrt{g(\eta)} K(x_2, \eta) \\ &\quad \times \left(\int_0^l F(\xi) \sqrt{g(\xi)} Y_n(\xi) d\xi \right) Y_n(\eta) d\eta. \end{aligned} \quad (39)$$

According to Proposition 2.4, the number of eigenvalues is not finite, that means $\lambda_n \rightarrow \infty$ when $n \rightarrow \infty$, and further

$$\frac{\lambda_n}{\lambda_n - \omega^2} = \frac{1}{1 - \frac{\omega^2}{\lambda_n}} \rightarrow 1. \quad (40)$$

By view of $F(x_2) := \int_0^l K(x_2, \xi) q_0(\xi) d\xi$, we get

$$F(\xi) \sqrt{g(\xi)} = \int_0^l K(\xi, \eta) \sqrt{g(\xi)} q_0(\eta) d\eta = \int_0^l R(\xi, \eta) \frac{q_0(\eta)}{\sqrt{g(\eta)}} d\eta. \quad (41)$$

Then,

$$\frac{q_0(\eta)}{\sqrt{g(\eta)}} \in C([0, l]), \quad (42)$$

From (41) and (42) we obtain, that the following series

$$\sum_{n=1}^{\infty} \int_0^l F(\xi) \sqrt{g(\xi)} Y_n(\xi) d\xi Y_n(\eta) \quad (43)$$

is absolutely and uniformly convergent on $]0, l[$.

Further, in view of (39)–(42) and (24) we get, that

$$\begin{aligned} \sum_{n=1}^{\infty} S_n(x_2) &= \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n - \omega^2} \int_0^l \sqrt{g(\eta)} K(x_2, \eta) \\ &\times \left(\int_0^l F(\xi) \sqrt{g(\xi)} Y_n(\xi) d\xi \right) Y_n(\eta) d\eta \\ &= \int_0^l \sqrt{g(\eta)} K(x_2, \eta) \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n - \omega^2} \\ &\times \left(\int_0^l F(\xi) \sqrt{g(\xi)} Y_n(\xi) d\xi \right) Y_n(\eta) d\eta \end{aligned} \quad (44)$$

is also absolutely and uniformly convergent on $]0, l[$, i.e., $w_0(x_2) \in C([0, l])$.

Analogously, we obtain

$$\begin{aligned} w_0^{(i)}(x_2) &= F^{(i)} + \int_0^l \sqrt{g(\eta)} K_{x_2}^{(i)}(x_2, \eta) \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n - \omega^2} \\ &\times \left(\int_0^l F(\xi) \sqrt{g(\xi)} Y_n(\xi) d\xi \right) Y_n(\eta) d\eta, \quad i = 1, \dots, 4. \end{aligned} \quad (45)$$

Because of

$$\sqrt{g(\eta)} K_{x_2}^{(i)}(x_2, \eta) \in C([0, l] \times]0, l[),$$

we get $F^{(i)} \in C([0, l])$ and $w^{(i)} \in C([0, l])$, $i = 1, \dots, 4$. \diamond

Proposition 2.9 *If*

$$\frac{q_0(x_2)}{\sqrt{g(x_2)}} \text{ is a continuous function on } [0, l], \quad (46)$$

then

$$w_0(x_2) \in \begin{cases} C^1([0, l]), & \text{in case of Problems 1, 2, 5;} \\ C^1([0, l]) \cap C([0, l]), & \text{in case of Problems 3, 6;} \\ C^1([0, l]) \cap C([0, l]), & \text{in case of Problems 7, 8;} \\ C([0, l]), & \text{in case of Problem 9;} \\ C^1([0, l]), & \text{in case of Problem 10;} \\ C^1([0, l]), & \text{in case of Problem 4.} \end{cases} \quad (47)$$

Proof. If we replace in (12) $q(x_2)$ by $q_0(x_2)$, we get the solution of Problems 1-10 for $q(x_2)$ replaced by $q_0(x_2)$. Therefore,

$$\begin{aligned} F(x_2) &= \int_0^l K(x_2, \xi) q_0(\xi) d\xi \\ &\in \begin{cases} C^1([0, l]), & \text{in case of Problems 1, 2, 5, 9;} \\ C^1([0, l]) \cap C([0, l]), & \text{in case of Problems 3, 6;} \\ C^1([0, l]) \cap C([0, l]), & \text{in case of Problems 7, 8;} \\ C^1([0, l]), & \text{in case of Problem 10;} \\ C^1([0, l]), & \text{in case of Problem 4.} \end{cases} \end{aligned} \quad (48)$$

Since $\frac{q_0(x_2)}{\sqrt{g(x_2)}}$ is a continuous on $[0, l]$, in view of (41) and theorem 2.6 series (43) is absolutely and uniformly convergent on

$$\begin{cases} [0, l], & \text{in case of Problems 1 - 3, 5 - 9;} \\]0, l], & \text{in case of Problems 10;} \\ [0, l[, & \text{in case of Problems 4.} \end{cases}$$

Therefore, taking into account (24), we obtain that the

series (44) (i.e., $\sum_{n=1}^{\infty} S_n(x_2)$) is convergence absolutely and uniformly on

$$\begin{cases} [0, l], & \text{in case of Problems 1 - 3, 5 - 9;} \\]0, l], & \text{in case of Problems 10;} \\ [0, l[, & \text{in case of Problems 4.} \end{cases} \quad (49)$$

Analogously, from (25), we get that the series in the right hand side of (45) (for $i = 1$) convergent absolutely and uniformly on

$$\begin{cases} [0, l], & \text{in case of Problems 1, 2, 5;} \\]0, l], & \text{in case of Problems 7, 8, 10;} \\ [0, l[, & \text{in case of Problems 3, 4, 6.} \end{cases} \quad (50)$$

Since

$$w_0(x_2) = F(x_2) + \omega^2 \sum_{n=1}^{\infty} S_n(x_2),$$

according to (48), (49), (50) we have (47). \diamond

Similarly, we can prove that the following series

$$\begin{aligned} (D(x_2)w_0''(x_2))^{(i)} &= (D(x_2)F''(x_2))^{(i)} \\ &+ \omega^2 \sum_{n=1}^{\infty} \left[\frac{1}{\lambda_n - \omega^2} \int_0^l F(\xi) \sqrt{g(\xi)} Y_n(\xi) d\xi \right] \\ &\times (D(x_2)X_n''(x_2))^{(i)}, \quad i = 1, 2 \end{aligned}$$

are absolutely and uniformly convergent in $[0, l]$.

So, if (46) is fulfilled, we have proved that the formal solution (37) is a solution of (27) under BCs 1–10.

3. A Cusped Elastic Plate-Fluid Interaction Problem

Let us consider the problem of the interaction of a plate whose variable flexural rigidity is given by the equation (2) and of a flow of the fluid.

Let the flow of the fluid be independent of x_1 , parallel to the plane $0x_2x_3$, i.e. $v_1 \equiv 0$, and generating bending of the plate. Let at infinity, for pressure we have

$$p(x_2, x_3, t) \rightarrow p_{\infty}(t), \quad \text{when } |x| \rightarrow \infty, \quad (51)$$

and let for the velocity components conditions at infinity be either

$$v_2(x_2, x_3, t) = O(1), \quad v_3(x_2, x_3, t) \rightarrow v_{3\infty}(t), \quad (52)$$

or

$$v_j(x_2, x_3, t) = O(1), \quad j = 2, 3, \quad (53)$$

where $v := (v_2, v_3)$ is a velocity vector of the fluid, $p(x_2, x_3, t)$ is a pressure, and $v_{3\infty}(t)$, $p_{\infty}(t)$ are given functions.

In what follows we suppose that the plate is so thin that, we can assume: the fluid occupies the whole space R^3 but the middle plane Ω of the plate.

Let,

$$\begin{aligned} I &:= \{[0, l] \times \{0\}\}, \\ \Omega^f &:= \{x_1, x_2, x_3 : x_1 = 0, x := (x_2, x_3) \in \mathbf{R}^2 \setminus I\}, \\ v_2, v_3 &\in C^1(\Omega^f) \cap C^1(t > 0). \end{aligned}$$

Transmission conditions for $v_j(x_2, x_3, t)$ ($j = 2, 3$) we can write in the following form (see [12, 19, 25])

$$v_3(x_2, 0, t) = \frac{\partial w(x_2, t)}{\partial t}, \quad x_2 \in [0, l], \quad t \geq 0. \quad (54)$$

In case of a viscous fluid we add to (54) transmission condition for $v_2(x_2, x_3, t)$

$$v_2(x_2, 0, t) = 0, \quad x_2 \in [0, l], \quad t \geq 0. \quad (55)$$

Because of incompressibility we have

$$\operatorname{div} v(x_2, x_3, t) = 0, \quad (x_2, x_3) \in \Omega^f, \quad t \geq 0, \quad (56)$$

and (see e.g., [8], p.5)

$$\sigma_{jk}^f = -p\delta_{jk} + \mu \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} \right), \quad j, k = \text{const} = 2, 3, \quad (57)$$

where σ_{jk}^f is a stress tensor, μ is a coefficient of viscosity, δ_{jk} is Kroneker delta. In case of ideal fluid $\mu = 0$.

From (56) and (57) we obtain

$$\begin{aligned} \sigma_{33}^f(x_2, x_3, t) &= -p(x_2, x_3, t) + 2\mu \frac{\partial v_3(x_2, x_3, t)}{\partial x_3} \\ &= -p(x_2, x_3, t) - 2\mu \frac{\partial v_2(x_2, x_3, t)}{\partial x_2}. \end{aligned} \quad (58)$$

In virtue of (58) and (55) yields

$$\sigma_{33}^f(x_2, 0_{\pm}, t) = -p(x_2, 0_{\pm}, t).$$

Therefore, the transmission condition for p has the following form

$$-p(x_2, 0_+, t) + p(x_2, 0_-, t) = q(x_2, t), \quad x_2 \in]0, l[. \quad (59)$$

In case of harmonic vibration, we have

$$\begin{aligned} w(x_2, t) &= e^{i\omega t} w_0(x_2), \quad q(x_2, t) = e^{i\omega t} q_0(x_2), \quad (60) \\ p(x_2, x_3, t) &= e^{i\omega t} p_0(x_2, x_3), \\ u_2(x_2, x_3, t) &= e^{i\omega t} u_2^0(x_2, x_3), \quad u_3(x_2, x_3, t) = e^{i\omega t} u_3^0(x_2, x_3), \\ v_2(x_2, x_3, t) &= i\omega e^{i\omega t} v_2^0(x_2, x_3), \quad v_3(x_2, x_3, t) = i\omega e^{i\omega t} v_3^0(x_2, x_3), \\ p_{\infty}(t) &= e^{i\omega t} p_{\infty}^0, \quad v_{3\infty}(t) = i\omega e^{i\omega t} v_{3\infty}^0, \quad p_{\infty}^0, \quad v_{3\infty}^0 = \text{const}, \end{aligned} \quad (61)$$

where $\omega = \text{const} > 0$, $v_2 = u_{2,t}$, $v_3 = u_{3,t}$ ($u_1 = 0$, u_2 , u_3 are components of the displacement vector).

3.1. Case of an Ideal Fluid

In case of the potential motion of the flow there exists a complex function $\Phi = \psi + i\varphi \in C^2(\Omega^f) \cap C^1(t > 0)$ such that

$$\begin{aligned} \frac{\partial \varphi(x_2, x_3, t)}{\partial x_2} &= \frac{\partial \psi(x_2, x_3, t)}{\partial x_3} = v_2(x_2, x_3, t), \\ \frac{\partial \varphi(x_2, x_3, t)}{\partial x_3} &= -\frac{\partial \psi(x_2, x_3, t)}{\partial x_2} = v_3(x_2, x_3, t). \end{aligned} \quad (62)$$

The pressure is given by the formula

$$p(x_2, x_3, t) = \rho^f \left[\frac{v_\infty^2}{2} + \frac{p_\infty}{\rho^f} + \frac{\partial \varphi_\infty}{\partial t} - \frac{\partial \varphi}{\partial t} - \frac{1}{2}(v_2^2 + v_3^2) \right], \quad (63)$$

where H is the class of Hölder continuous functions.

$w(x_2, t)$ is given by the equation (1). Therefore, taking into account transmission condition (59), we have

$$\begin{aligned} (x_2^\alpha (l - x_2)^\beta w_{,22}(x_2, t))_{,22} = & -\frac{2h(x_2)\rho^s}{D_0} w_{,tt}(x_2, t) \\ & + \frac{-p\left(x_2, \overset{(+)}{h}(x_2), t\right) + p\left(x_2, \overset{(-)}{h}(x_2), t\right)}{D_0}. \end{aligned} \quad (64)$$

For $\Phi_{,2}(x_2, x_3, t) = -v_3 + iv_2$, in view of (54) and (52), we get the following expression (see [17])

$$\begin{aligned} \Phi_{,2} = & -\frac{1}{\pi i \sqrt{(x_2 + ix_3)(x_2 + ix_3 - l)}} \\ & \times \int_0^l \frac{\sqrt{(\xi_2 + ix_3)(\xi_2 + ix_3 - l)}}{(\xi_2 - x_2) - ix_3} w_{,t}(\xi_2, t) d\xi_2 \\ & + v_{3\infty} \frac{x_2 + ix_3 - l/2}{\sqrt{(x_2 + ix_3)(x_2 + ix_3 - l)}}. \end{aligned} \quad (65)$$

If we consider harmonic vibration, then taking into account (60) and (61) we obtain

$$\varphi(x_2, x_3, t) = ie^{i\omega t} \varphi_0(x_2, x_3), \quad \psi(x_2, x_3, t) = ie^{i\omega t} \psi_0(x_2, x_3).$$

From (65) we have an expression for v_3 . By means of the latter, in view of (62), we can calculate φ which we have to substitute in (63). Then substituting the obtained expression of $p(x_2, x_3, t)$ in (63), by virtue of (60), we get the following expression for $q_0(x_2)$

$$\begin{aligned} q_0(x_2) = & \frac{\omega^2 \rho^f}{\pi} \int_0^l w_0(\xi) \int_{-\overset{(+)}{h}(x_2)}^{\overset{(+)}{h}(x_2)} \frac{\sqrt{r(\xi, x_3)}}{\sqrt{r(x_2, x_3)}} \\ & \times \frac{(x_2 - \xi) \cos \frac{\phi(\xi, x_3) - \phi(x_2, x_3)}{2} + x_3 \sin \frac{\phi(\xi, x_3) - \phi(x_2, x_3)}{2}}{(\xi - x_2)^2 + x_3^2} dx_3 d\xi \\ & - \omega^2 \rho^f \int_{-\overset{(+)}{h}(x_2)}^{\overset{(+)}{h}(x_2)} \left\{ \left(x_2 - \frac{l}{2} \right) \cos \frac{\phi(x_2, x_3)}{2} + x_3 \sin \frac{\phi(x_2, x_3)}{2} \right\} \frac{v_{3\infty}^0 dx_3}{\sqrt{r(x_2, x_3)}}, \end{aligned} \quad (66)$$

where $\phi(x_2, x_3)$ is defined either by

$$\cos \phi(x_2, x_3) = (x_2^2 - x_3^2 - lx_2)/r(x_2, x_3)$$

or by

$$\sin \phi(x_2, x_3) = (2x_2 - l)x_3/r(x_2, x_3)$$

and

$$r(x_2, x_3) = \sqrt{(x_2^2 - x_3^2 - lx_2)^2 + ((2x_2 - l)x_3)^2}.$$

Taking into account (60), (61), (66), from (64) after integrating four times with respect to x_2 we get the following relation

$$\begin{aligned} w_0(x_2) &= 2\rho^s \omega^2 \int_{x_2^0}^{x_2} h(\xi) K(x_2, \xi) w_0(\xi) d\xi = \int_{x_2^0}^{x_2} (c_1 \xi + c_2) (x_2 - \xi) D^{-1}(\xi) d\xi \\ &- c_3 x_2 + c_4 + \int_{x_2^0}^{x_2} K(x_2, \xi) q_0(\xi) d\xi, \end{aligned} \quad (67)$$

where

$$x_2^0 \in]0, l[, \quad K(x_2, \xi) = - \int_{\xi}^{x_2} (x_2 - \eta)(\xi - \eta) D^{-1}(\eta) d\eta.$$

Constants c_i ($i = 1, \dots, 4$) should be defined from the admissible boundary value conditions (see in Section 2, Problems 1-10).

Let us consider, e.g., Problem 8. Then for $w_0(x_2)$ we get the following equation

$$\begin{aligned} w_0(x_2) &= \omega^2 \int_0^l K_1(x_2, \xi) w_0(\xi) d\xi \\ &- 2\rho^s \omega^2 \left\{ \int_{x_2^0}^{x_2} h(\xi) K(x_2, \xi) w_0(\xi) d\xi + \int_{x_2^0}^l h(\xi) K_l(x_2, \xi) w_0(\xi) d\xi \right. \\ &\left. + \int_0^{x_2^0} h(\xi) K_0(x_2, \xi) w_0(\xi) d\xi \right\} = f(x_2), \end{aligned} \quad (68)$$

where

$$\begin{aligned} K_0(x_2, \xi) &= \xi \left\{ \int_l^{x_2} x_2 D^{-1}(\eta) d\eta - \int_0^{x_2} \eta D^{-1}(\eta) d\eta \right\} - K(0, \xi), \\ K_l(x_2, \xi) &= x_2 \int_l^{x_2} \eta D^{-1}(\eta) d\eta \end{aligned}$$

$$-\int_0^{x_2} \eta^2 D^{-1}(\eta) d\eta + x_2 \int_{\xi}^l (\eta - \xi) D^{-1}(\eta) d\eta,$$

$$\begin{aligned} K_1(x_2, \xi) = & \frac{\rho^f}{\pi} \left\{ \int_{x_2^0}^l K_l(x_2, \zeta) \int_{-h^{(+)}(\zeta)}^{h^{(+)}(\zeta)} \frac{\sqrt{r(\xi, x_3)}}{\sqrt{r(\zeta, x_3)}} \right. \\ & \times \frac{(\zeta - \xi) \cos \frac{\phi(\xi, x_3) - \phi(\zeta, x_3)}{2} + x_3 \sin \frac{\phi(\xi, x_3) - \phi(\zeta, x_3)}{2}}{(\xi - \zeta)^2 + x_3^2} dx_3 d\zeta \\ & + \int_{x_2^0}^0 K_0(x_2, \zeta) \int_{-h^{(+)}(\zeta)}^{h^{(+)}(\zeta)} \frac{\sqrt{r(\xi, x_3)}}{\sqrt{r(\zeta, x_3)}} \\ & \times \frac{(\zeta - \xi) \cos \frac{\phi(\xi, x_3) - \phi(\zeta, x_3)}{2} + x_3 \sin \frac{\phi(\xi, x_3) - \phi(\zeta, x_3)}{2}}{(\xi - \zeta)^2 + x_3^2} dx_3 d\zeta \\ & + \int_{x_2^0}^{x_2} K(x_2, \zeta) \int_{-h^{(+)}(\zeta)}^{h^{(+)}(\zeta)} \frac{\sqrt{r(\xi, x_3)}}{\sqrt{r(\zeta, x_3)}} \\ & \times \left. \frac{(\zeta - \xi) \cos \frac{\phi(\xi, x_3) - \phi(\zeta, x_3)}{2} + x_3 \sin \frac{\phi(\xi, x_3) - \phi(\zeta, x_3)}{2}}{(\xi - \zeta)^2 + x_3^2} dx_3 d\zeta \right\}; \end{aligned}$$

$$f(x_2) = x_2 \left(g_{22} + h_{22} \int_{x_2^0}^l \xi D^{-1}(\xi) d\xi + h_{11} \int_{x_2^0}^l D^{-1}(\xi) d\xi \right)$$

$$+ g_{11} + h_{22} \int_0^{x_2^0} \xi^2 D^{-1}(\xi) d\xi$$

$$- h_{11} \int_0^{x_2^0} \xi D^{-1}(\xi) d\xi - \int_{x_2^0}^{x_2} (h_{22} \xi + h_{11})(x_2 - \xi) D^{-1}(\xi) d\xi - \omega^2 \rho^f \left\{ \int_{x_2^0}^l K_l(x_2, \xi) \right.$$

$$\times \int_{-h^{(+)}(\xi)}^{h^{(+)}(\xi)} \left\{ \left(\xi - \frac{l}{2} \right) \cos \frac{\phi(\xi, x_3)}{2} + x_3 \sin \frac{\phi(\xi, x_3)}{2} \right\} \frac{v_{3\infty}^0 dx_3}{\sqrt{r(\xi, x_3)}} d\xi$$

$$\begin{aligned}
& - \int_{x_2^0}^0 K_0(x_2, \xi) \int_{-h^{(+)}(\xi)}^{h^{(+)}(\xi)} \left\{ \left(\xi - \frac{l}{2} \right) \cos \frac{\phi(\xi, x_3)}{2} + x_3 \sin \frac{\phi(\xi, x_3)}{2} \right\} \\
& \times \frac{v_{3\infty}^0 dx_3}{\sqrt{r(\xi, x_3)}} d\xi - \int_{x_2^0}^{x_2} K_0(x_2, \xi) \\
& \times \int_{-h^{(+)}(x_2)}^{h^{(+)}(x_2)} \left\{ \left(\xi - \frac{l}{2} \right) \cos \frac{\phi(\xi, x_3)}{2} + x_3 \sin \frac{\phi(\xi, x_3)}{2} \right\} \frac{v_{3\infty}^0 dx_3}{\sqrt{r(\xi, x_3)}} d\xi \Bigg\}.
\end{aligned}$$

It is easy to show that

$$2\rho^s h(\xi) K(x_2, \xi),$$

$$2\rho^s h(\xi) K_0(x_2, \xi),$$

$$2\rho^s h(\xi) K_l(x_2, \xi),$$

$$K_1(x_2, \xi) \in C([0, l])$$

(in our case $0 \leq \alpha < 2$, $0 \leq \beta < 1$). The integral equation (68) can be solved by method of successive approximations.

Remark 3.1 In case of the other above boundary conditions (see Problems 1-7, 9, 10), the problem under consideration is solved analogously and in all the cases we get integral equations of (68) type.

Thus, the following Proposition is valid.

Proposition 3.2 Problem of the harmonic vibration of the plate with two cusped edges under action of the incompressible ideal fluid [i.e., equations (62), (63), (64), under transmission conditions (54), (59) and under conditions at infinity (51), (52) and BCs (see Problems 1-10)] has a unique solution when

$$\omega^2 < \frac{1}{Ml},$$

where

$$\begin{aligned}
M := \max_{x_2, \xi \in [0, l]} \{ & |2\rho^s h(\xi) K(x_2, \xi)|, \\
& |2\rho^s h(\xi) K_0(x_2, \xi)|, |2\rho^s h(\xi) K_l(x_2, \xi)|, |K_1(x_2, \xi)| \}.
\end{aligned}$$

3.2. Case of a Viscous Fluid

Let the motion of the fluid be sufficiently slow, i.e., v_j and $v_{j,k}$ ($i, k = 2, 3$) be so small that linearization of Navier-Stokes equations (see [12, 19, 25]) be admissible. Hence,

$$\begin{aligned}\frac{\partial v_2}{\partial t} &= -\frac{1}{\rho^f} \frac{\partial p}{\partial x_2} + \nu \Delta v_2 + F_2(x_2, x_3, t), \\ \frac{\partial v_3}{\partial t} &= -\frac{1}{\rho^f} \frac{\partial p}{\partial x_3} + \nu \Delta v_3 + F_3(x_2, x_3, t),\end{aligned}\tag{69}$$

where $\nu = \mu/\rho^f$, $\Delta = \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, $F := (F_2, F_3)$ is a volume force. Let

$$\begin{aligned}v_i &\in C^2(\Omega^f) \cap C(\mathbf{R}^2) \cap C(t > 0), \quad i = 2, 3; \\ p &\in C^2(\Omega^f); \\ q, 2(\cdot, t) &\in H([0, l]),\end{aligned}$$

and

$$F_i \in C^2(\Omega^f), \quad i = 2, 3.$$

Let

$$A^\infty := \lim_{|x| \rightarrow \infty} \left(\int_0^{x_2} F_2(\xi_2, x_3) d\xi_2 + \int_0^{x_3} F_3(0, \xi_3) d\xi_3 \right).$$

After differentiation of the first equation of (69) with respect to x_2 , of the second equation of (69) with respect to x_3 and termwise summation, by virtue of (56), we obtain that $p(x_2, x_3, t)$ is satisfying the following equation

$$\Delta p(x_2, x_3, t) = \left(\frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right) \rho^f.\tag{70}$$

In case of harmonic vibration in the fluid part, from (69), (61), (70) we obtain the following system

$$\Delta p_0(x_2, x_3) = \rho^f \left(\frac{\partial F_2^0}{\partial x_2} + \frac{\partial F_3^0}{\partial x_3} \right),\tag{71}$$

$$-\omega^2 u_j^0 = -\frac{1}{\rho^f} \frac{\partial p_0}{\partial x_j} + \nu i \omega \Delta u_j^0 + F_j^0(x_2, x_3), \quad j = 2, 3,\tag{72}$$

where $F_j(x_2, x_3) = e^{i\omega t} F_j^0(x_2, x_3)$.

The transmission conditions (59), (54), (55) and conditions at infinity (51) and (53) have the following forms

$$-p_0^+(x_2) + p_0^-(x_2) = q_0(x_2), \quad x_2 \in]0, l[, \tag{73}$$

$$u_3^0(x_2, 0) = w_0(x_2), \quad u_2^0 = 0, \quad x_2 \in [0, l], \tag{74}$$

$$p_0|_{|x| \rightarrow \infty} = p_0^\infty, \quad u_j^0|_{|x| \rightarrow \infty} = O(1), \quad j = 2, 3. \tag{75}$$

After taking apart real and imaginary parts in (72) we have

$$u_j^0 = \frac{1}{\omega^2 \rho^f} \frac{\partial p_0}{\partial x_j} - \frac{1}{\omega^2} F_j^0(x_2, x_3), \quad j = 2, 3, \quad (76)$$

$$\Delta u_j^0 = 0, \quad j = 2, 3. \quad (77)$$

Therefore, taking into account (71),

$$-\frac{\partial}{\partial x_j} \left(\frac{\partial F_2^0}{\partial x_2} + \frac{\partial F_3^0}{\partial x_3} \right) + \Delta F_j^0 = 0, \quad j = 2, 3. \quad (78)$$

In view of (70) and (78) for $j=2$, we have

$$\begin{aligned} \Delta p(x_2, x_3) &= \rho^f \left(\int_0^{x_2} \Delta F_2(\xi_2, x_3) d\xi_2 + a(x_3) \right) \\ &= \left(\frac{\partial F_2}{\partial x_2}(x_2, x_3) + \frac{\partial F_3}{\partial x_3}(x_2, x_3) \right) \rho^f \\ &\quad - \left(\frac{\partial F_2}{\partial x_2}(0, x_3) + \frac{\partial F_3}{\partial x_3}(0, x_3) \right) \rho^f + a(x_3) \rho^f. \end{aligned}$$

By virtue of (70) we get

$$a(x_3) = \frac{\partial F_2}{\partial x_2}(0, x_3) + \frac{\partial F_3}{\partial x_3}(0, x_3).$$

On the other hand

$$\int_0^{x_2} \Delta F_2 d\xi_2 = \Delta \int_0^{x_2} F_2 d\xi_2 - \frac{\partial F_2}{\partial x_2}(0, x_3).$$

Taking into last tree expressions we obtain

$$\Delta \left(p_0 - \rho^f \int_0^{x_2} F_2(\xi_2, x_3) d\xi_2 \right) = \rho^f \frac{\partial F_3}{\partial x_3}(0, x_3).$$

The solution of the last equation under condition (73), (75), in case $q_0(x_2) \in H([0, l])$ (H is the class of Hölder continuous functions), has the following form (see [17])

$$\begin{aligned} p_0(x_2, x_3) &= -\frac{x_3}{2\pi} \int_0^l \frac{q_0(\xi_2) d\xi_2}{(\xi_2 - x_2)^2 + x_3^2} + \rho^f \int_0^{x_2} F_2(\xi_2, x_3) d\xi_2 \\ &\quad + \rho^f \int_0^{x_3} F_3(0, \xi_3) d\xi_3 + p_0^\infty - \rho^f A^\infty. \end{aligned} \quad (79)$$

Substituting (79) into (76), for u_3^0 we get

$$\begin{aligned} u_3^0 = & \frac{1}{2\pi\omega^2\rho^f} \int_0^l \frac{q_0(\xi)[x_3^2 - (\xi - x_2)^2]}{[(\xi - x_2)^2 + x_3^2]^2} d\xi + \frac{1}{\omega^2} \left(\int_0^{x_2} \frac{\partial F_2}{\partial x_3}(\xi_2, x_3) d\xi_2 \right. \\ & \left. + \int_0^{x_3} F_3(0, \xi_3) d\xi_3 \right) - \frac{1}{\omega^2} F_3^0(x_2, x_3). \end{aligned} \quad (80)$$

Hence, using transmission condition (74) for u_3^0 , we get the following expression

$$\begin{aligned} w_0(x_2) = & -\frac{1}{2\pi\omega^2\rho^f} \int_0^l \frac{q_0(\xi)}{(\xi - x_2)^2} d\xi + \\ & + \frac{1}{\omega^2} \left(\int_0^{x_2} \frac{\partial F_2}{\partial x_3}(\xi_2, 0) d\xi_2 F_3(0, 0) \right) \frac{1}{\omega^2} F_3^0(x_2, 0), \end{aligned} \quad (81)$$

where $x_2 \in [0, l]$ and the super singular integral on the right hand side we define as H'adamard integral (see [1, 3]).

Substituting (81) into (28), for $q_0(x_2)$ we obtain the following supersingular integral equation

$$\begin{aligned} & \int_0^l \frac{q_0(\xi)}{(\xi - x_2)^2} d\xi + 2\pi\omega^2\rho^f \int_0^l K(x_2, \xi) q_0(\xi) d\xi \\ & - \omega^2 \int_0^l \left\{ \int_0^l \frac{K(x_2, \xi) g(\xi)}{(\eta - \xi)^2} d\xi \right\} q_0(\eta) d\eta \\ & = -2\pi\rho^f \left[F_3(x_2, 0) - \omega^2 \int_0^l K(x_2, \xi) g(\xi) F_3(\xi, 0) d\xi \right] \\ & + 2\pi\rho^f \left[\int_0^{x_2} \frac{\partial F_2}{\partial x_3}(\xi, 0) d\xi + F_3(0, 0) \right. \\ & \left. - \omega^2 \int_0^l K(x_2, \xi) g(\xi) \left\{ \int_0^\xi \frac{\partial F_2}{\partial x_3}(\eta, 0) + F_3(0, 0) \right\} d\eta d\xi \right] =: f(x_2). \end{aligned} \quad (82)$$

We will find approximate solution of (82) using the method of solving given in books [1, 3] for $q'_0(x_2) := (dq_0(x_2)/dx_2) \in H([0, l])$.

Let divide interval $[0, l]$ into N parts as follows

$$y'_N := \frac{lk}{N}, \quad k = \overline{0, N}, \quad y_k := \frac{lk}{N} + \frac{l}{2N}, \quad k = \overline{0, N-1},$$

$$q_{0N} := (q_0(y_0), \dots, q_0(y_{N-1})),$$

we will call q_{0N} approximate solution of (82). For q_{0N} we get the following system of linear equations

$$\begin{aligned}
 & -\frac{4N}{l}q_0(y_i) - \sum_{\substack{j=0 \\ j \neq i}}^{N-1} q_0(y_j) \left[\frac{1}{y'_{j+i} - y_i} - \frac{1}{y'_j - y_i} \right] \\
 & + \frac{2\pi\omega^2\rho^f l}{N} \sum_{j=0}^{N-1} K(y_i, y_j)g(y_j)q_0(y_j) - \frac{\omega^2 l}{N} \sum_{j=0}^{N-1} K(y_i, y_j)g(y_j) \\
 & \times \left\{ \frac{4N}{l}q_0(y_j) + \sum_{\substack{j=0 \\ k \neq j}}^{N-1} q_0(y_k) \left[\frac{1}{y'_{k+j} - y_j} - \frac{1}{y'_k - y_j} \right] \right\}. \quad (83) \\
 & = f(y_i), \quad i = \overline{0, N-1}.
 \end{aligned}$$

It is well-known (see [3]) that the determinant of the system (83) is not zero. Therefore, (83) is uniquely solvable.

Now, we have to estimate the error of the approximate solution of the equation (82). Let us denote by q_0^* the solution of (82), by q_{0N}^* the solution of (83) and let \hat{q}_{0N}^* be a projection of q_0^* on y_k . Further, we obtain

$$\begin{aligned}
 & \left| -\frac{4N}{l} (q_{0N}^*(y_i) - \hat{q}_{0N}^*(y_i)) - \sum_{\substack{j=0 \\ j \neq i}}^{N-1} \{q_{0N}^*(y_j) - \hat{q}_{0N}^*(y_j)\} \left\{ \frac{1}{y'_{j+i} - y_i} - \frac{1}{y'_j - y_i} \right\} \right| \\
 & = \left| \int_0^l \frac{q_0^*(\xi)}{(\xi - y_i)^2} d\xi - \frac{4N}{l} \hat{q}_{0N}^*(y_i) + \sum_{\substack{j=0 \\ j \neq i}}^{N-1} \hat{q}_{0N}^*(y_j) \left\{ \frac{1}{y'_{j+i} - y_i} - \frac{1}{y'_j - y_i} \right\} \right| \\
 & \leq \left| \int_{y'_i}^{y'_{i+1}} \frac{q_0^*(\xi) - q_0^*(y_i)}{(\xi - y_i)^2} d\xi \right| + \sum_{\substack{j=0 \\ j \neq i}}^{N-1} \left| \int_{y'_i}^{y'_{i+1}} \frac{q_0^*(\xi) - q_0^*(y_j)}{(\xi - y_j)^2} d\xi \right| =: I_1 + I_2.
 \end{aligned}$$

Therefore, since $q_0'(x_2) \in H([0, l])$, we have that there exist $A = \text{const} > 0$, and $\alpha_1 = \text{const } 0 < \alpha_1 < 1$ such that

$$|q_0'(y_1) - q_0'(y_2)| \leq A|y_1 - y_2|^{\alpha_1}.$$

Using the following expression

$$\int_{y'_i}^{y'_{i+1}} \frac{d\xi}{\xi - y_i} = \ln|\xi - y_i|_{y'_i}^{y'_{i+1}} = \ln \frac{l/(2N)}{l/(2N)} = 0,$$

we obtain

$$\begin{aligned}
 I_1 &= \left| \int_{y'_i}^{y'_{i+1}} \frac{q_0^*(\xi) - q_0^*(y_i) - (\xi - y_i) \left\{ \frac{dq_0^*(\xi)}{d\xi} \Big|_{\xi=y_i} \right\}}{(\xi - y_i)^2} d\xi \right| \\
 &= \left| \int_{y'_i}^{y'_{i+1}} \frac{\frac{dq_0^*(\xi)}{d\xi} - \frac{dq_0^*(\xi)}{d\xi} \Big|_{\xi=y_i}}{\xi - y_i} d\xi \right| \leq A \left(\frac{2N}{l} \right)^{-\alpha_1}. \quad (84)
 \end{aligned}$$

Analogously, we get

$$I_2 \leq A(N-1) \left(\frac{2N}{l} \right)^{-\alpha_1}. \quad (85)$$

From (84) and (85) we obtain that the error of this method might be too large. For getting the most better results instead of the system (83) we consider the following system

$$\begin{aligned}
 &a_{ii}q_0(y_i) - \sum_{j=0}^{N-1} q_0(y_j) \left[\frac{1}{y'_{j+i} - y_i} - \frac{1}{y'_j - y_i} \right] \\
 &+ \frac{2\pi\omega^2\rho^f l}{N} \sum_{j=0}^{N-1} K(y_i, y_j)g(y_j)q_0(y_j) + \frac{\omega^2 l}{N} \sum_{j=0}^{N-1} K(y_i, y_j)g(y_j) \\
 &\times \left\{ a_{ii}q_0(y_j) - \sum_{j=0}^{N-1} q_0(y_k) \left[\frac{1}{y'_{k+j} - y_j} - \frac{1}{y'_k - y_j} \right] \right\}. \quad (86) \\
 &= f(y_i), \quad i = \overline{0, N-1}.
 \end{aligned}$$

where

$$\begin{aligned}
 a_{ii} &:= -\frac{4N}{l} \int_{\Delta_{ii}} \frac{d\xi}{(\xi - y_i)^2}, \quad \Delta_{ii} := [0, l] \cap \left[y'_i - \frac{n}{N}, y'_{i+1} + \frac{n}{N} \right], \\
 n &:= \sqrt{N} \quad \sum' := \sum_{\substack{j=0 \\ j \neq i-1, i, i+1}}^{N-1}.
 \end{aligned}$$

After repeating above calculation we get

$$|q_0^* - q_{0N}^*| \leq A \left(\frac{2n}{l} \right)^{-\alpha_1},$$

where q_0^* and q_{0N}^* are the solutions of the equations (82) and (86) respectively.

After calculating q_{0N} , from (79) and (81) we get approximate expressions for $p_0(x_2, x_3)$ and $w_0(x_2)$, as follows

$$p_0(x_2, x_3) = -\frac{x_3 l}{2\pi N} \sum_{j=0}^{N-1} \frac{q_0(y_j)}{(y_j - x_2)^2 + x_3^2} + \rho^f \int_0^{x_2} F_2(\xi_2, x_3) d\xi_2$$

$$\begin{aligned}
& + \rho^f \int_0^{x_3} F_3(0, \xi_3) d\xi_3 + p_0^\infty + A^\infty, \quad (x_2, x_3) \in \Omega^f; \\
w_0(y_i) &= -\frac{1}{2\pi\omega^2\rho^f} \left\{ a_{ii}q_0(y_i) - \sum_{j=0}^{N-1} q_0(y_j) \left[\frac{1}{y'_{j+i} - y_i} - \frac{1}{y'_j - y_i} \right] \right\} \\
& + \frac{1}{\omega^2} \left(\int_0^{y_i} \frac{\partial F_2}{\partial x_3}(\xi_2, 0) d\xi_2 + F_3(0, 0) \right) - \frac{1}{\omega^2} F_3^0(y_i, 0), \quad x_2 \in [0, l],
\end{aligned}$$

Let us denote by $\bar{w}_0(y_i)$ the projection of w_0 on y_i and let estimate the error of the approximate solution of deflection. If the repeat the above calculation we get

$$|\bar{w}_0(y_i) - w_0(y_i)| \leq A \frac{1}{2\rho^f\pi\omega^2} \left(\frac{2n}{l} \right)^{-\alpha_1}.$$

Further, after Substituting $p_0(x_2, x_3)$ in (76) we obtain $u_j^0(x_2, x_3)$.

Proposition 3.3 *In case of the harmonic vibration of the plate with two cespded edges under action of the incompressible viscous fluid [i.e, equations (71), (72), (27) under transmission conditions (73), (74), conditions at infinity (75) and BCs (see Problems 1-10)] all quantities ($u_2^0(x_2, x_3)$, $u_3^0(x_2, x_3)$, $p_0(x_2, x_3)$ and $w_0(x_2)$) can be expressed by lateral load ($q_0(x_2)$) (see formulas (79)-(81)) and for the calculating of $q_0(x_2)$ we get (82) type super singular integral equation, where super singular integral is defined as H' adamard integral. This equation has solutions in the class $q'_0 \in H([0, l])$ (H is a class of Hölder continuous functions).*

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