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The Exact Solution of Nonlinear Stress-Free Convection Under the Influence of Magnetic Fields

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This paper aims at transforming nonlinear equations (Navier-Stokes, energy and magnetic induction) in steady state into ordinary high order differential equations. The exact solution for it has been found using a suitable transformation. The second goal is studying the state conditions of heating from below in a horizontal layer and studying the influence of the magnetic field on the phenomenon of convection itself. This model paper has uncovered the properties of electricity-conducting fluid elements. These properties have been disclosed neither in theory nor in practice. This study has been applied on the stress-free boundaries. All of which have realistic manifestation, in nature. They study convection in geophysics and astrophysics. Results have been illustrated in three dimensions to generalize the study and use it practically.

Keywords: convection, solar, magnetic field, stress-free, exact solution, nonlinear.

1. Introduction

Every feature on the sun that is associated with solar variability is created or influenced by magnetic fields. Thus, they are a natural candidate for modulating the structure and energy output of the sun on a more global scale, that is the scale of the entire sun itself. Since energy is transported by convection (physical motion of the gases) in the outer regions of the sun the way in which magnetic fields modify the convective flows vary considerably (due to the physical effects of stratification, partial ionization, radiation, and the like), changes in the uppermost layers are seen at the surface first, then those in the deeper layer, and finally those which occur at the position in the sun which marks the interface between where energy is transported by convection (above) and by radiation (below). This transition zone is about seventeenths of the way from the sun's center to the surface [11]. In fact, the sun has a cycle that repeats half every 11 years. During which the structure of the magnetic field dramatically varies. At the beginning of the cycle, the line of magnetic force runs north and south between the sun's magnetic poles. This is the period of minimum magnetic activity called the "Solar Minimum". However, this condition does not last.

As the sun rotates, the convection zone spins faster at the equator than it does at the poles. Beneath the convection zone, the radiation zone spins as a solid mass. The different way that these two zones move causes the sun's magnetic field to stretch at the equator. As the solar cycle continues. these lines of magnetic force continue to stretch. Like a rubber-band that is twisted too much, the magnetic field begins to buckle. Eventually, the magnetic force, which is generated beneath the convection zone, breaks the surface of the sun [4,8,10]. These phenomena are like giant magnetic storm, that not only alter the sun's surface, but also eject out powerful bursts of energy into the solar system. The peak of all this activity is called the "Solar Maximum". At these times we can on Earth experience magnetic disturbances like disruptions in satellite communications and atmospheric events like the Aurora Borealis. Following the Solar Maximum, the magnetic field begins to unwind and activity on the sun subsides. Gradually, the sun returns to the Solar Minimum and the cycle begins again. The sun reached a solar Minimum in the mid 1990's. The next solar Maximum occurred around 2003 [1,2,14,15].

This paper starts with a short description of mathematical problem and introduces the main method of the analysis; the boundaries are stress-free problem in Section 2. The general steady solution is presented in Section 3. Discussion is given in a concluding remarks.

2. Mathematical formulation

We consider a horizontal fluid layer of thickness d, which is permeated by a homogeneous vertical magnetic field with flux density B. The electrical conductivity of the fluid is so high that the magnetic field tends to impede the motions of the magnetic diffusivity λ is not low enough for the magnetic Reynolds number to reach values of the order unity or larger. This situation is realized in typical laboratory experiments with liquid metals. It also applies to processes in planetary cores of sufficiently small scale.

In order to introduce a dimensionless description of the problem we use d as length scale, $\frac{d^2}{\kappa}$ as time scale, $\frac{\kappa\nu}{d^3\gamma g}$ as the scale of the temperature and $\frac{B\kappa}{\lambda}$ as the scale for the magnetic field where the symbols κ , λ , ν , γ , g refer to the thermal diffusivity, the magnetic diffusivity, the kinematic viscosity, the coefficient of thermal expansion and the acceleration of gravity, respectively. The equations of motion in the Boussinesq approximation, the heat equation for the deviation θ of the temperature from the static distribution and the equation of magnetic induction in the magneto-hydrodynamic approximation are

$$P^{-1}\left(\frac{\partial}{\partial t} + u \cdot \nabla\right) u = -\nabla\pi + k\theta + \nabla^2 u + Q\frac{\kappa}{\lambda}B \cdot \nabla B, \qquad (1)$$

$$\nabla . u = 0, \quad \nabla . B = 0, \qquad (2)$$

$$\left(\frac{\partial}{\partial t} + u \,.\, \nabla\right)\theta = Rk \,.\, u + \nabla^2\theta \tag{3}$$

$$\frac{\kappa}{\lambda} \left(\frac{\partial}{\partial t} + u \, \cdot \, \nabla \right) B = \frac{\kappa}{\lambda} B \, \cdot \, \nabla u + \nabla^2 B \,, \tag{4}$$

El-Kholy, SA

where the Rayleigh number R, the Prandtl number P and the Chandrasekhar number Q are defined respectively by

$$R = \gamma (T_2 - T_1) \frac{gd^3}{\kappa \nu}, \quad P = \frac{\nu}{\kappa}, \quad Q = \frac{d^2}{\rho \mu \lambda}.$$
 (5)

The temperatures T_1 and T_2 with $T_1 < T_2$ are prescribed at the upper and the lower stress-free boundaries of layer: ρ and μ denote the density fluid and the magnetic permeability of the fluid and unit vector k is directed opposite to the direction of gravity. We shall use a Cartesian system of coordinates with the z-coordinate in the direction of k and the x-coordinate in the direction of the axis of the convection rolls [15]. We assume that magnetic field is uniform in this case and parallel to k.

We describe the state of convection us a deviation from the static state. Using d as the length scale we introduce a non-dimensional cartesian co-ordinates with the z-coordinate in the direction of k and origin at the bottom of the layer. Restricting our attention to the case of two-dimensional steady convection we use non-dimensional variables ϕ , θ and g to write the following equations [6,14]

$$u = \left(\frac{\partial\phi}{\partial z}, 0, -\frac{\partial\phi}{\partial x}\right) \tag{6}$$

$$B = \left(\frac{\partial g}{\partial z}, 0, -1 - \frac{\partial g}{\partial x}\right)$$

$$k \cdot \nabla \Delta_2 g \approx \Delta_2 \phi \qquad (7)$$

$$\Delta_2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)$$

We consider the function $\phi(x, z)$, $\theta(x, z)$ and g(x, z) as the form [9,10,12]

$$f(x,z) = x + f(z).$$
(8)

Applying the curl operator to equation (1) and substituting the assumption (6)-(8) into (1)-(4) we have four partial differential equations. By using the assumption (8), these partial differential equations are reduced to the following two linear differential equations

$$\frac{d^4\phi(z)}{dz^4} + p^{-1}\frac{d^3\phi(z)}{dz^3} + 2Q\frac{d^2\phi(z)}{dz^2} = 0, \qquad (9)$$

$$\frac{d^2\theta(z)}{dz^2} + \frac{d\theta(z)}{dz} = R + \frac{d\phi(z)}{dz}, \qquad (10)$$

with the stress-free boundary conditions as following

$$\phi(z) = \frac{d^2\phi(z)}{dz^2} = 0, \text{ at } Z = \pm \frac{1}{2},$$
(11)

$$\theta(z) = 0$$
, at $Z = \pm \frac{1}{2}$. (12)

As [3,7,13] we can easily have analytical solution in the form as shown in the Appendix.

3. The steady solution

The heat transfer is usually represented in terms of the Nusselt number which describes the ratio between the heat transport which convection and what would be without convection at a given Rayleigh number. In describing an analytical result for steady convection rolls we shall concentrate on the convective heat transfer. Which not only is the parameter of primary physical interest, but also appears to characterize best the other aspects of convection is viscous dissipation occur at the same rate is the convection heat transport. Number of analytical studies of convection rolls have handed results for the Nusselt number [1,2,6]

$$Nu = 1 + \frac{\langle u \, . \, k\theta(z) \rangle}{R} \tag{13}$$

4. Discussion

We present results from direct simulations of turbulent compressible hydromagnetic convection above a stable overshoot layer. Spontaneous dynamo action occurs followed by saturation, with most of the generated magnetic field appearing as coherent flux tubes in the vicinity of strong downdrafts, where both the generation and destruction of magnetic field is most vigorous. Whether or not this field is amplifield depends on the sizes of the magnetic Reynolds and magnetic Prandtl numbers. Joule dissipation is balanced mainly by the work done against the magnetic curvature force. It's curvature force, which is also responsible for the saturation of the dynamo.



Figure 1 Relation between the velocity of the particles of fluid in the depth of layer with difference of Prandtl numbers at Q = 10

Figure 1 shows the velocity of the particles during convection. The values of the velocity of the particle between maximum and minimum values are symmetric. We notice that more turbulence at weak magnetic value Q = 10 and small value of Prandtl number than otherwise. We also notice that the wavely turbulence of Benard's cell in the bottom layer is greater than that at the top.

16



Figure 2 Relations between the velocity of the particles of fluid in the depth of layer with difference of Prandtl numbers at Q = 80

Figure 2 shows the growth of turbulence with the increase of magnetic field as high as Q = 80 especially at very low of Prandtl number $P \leq 2.25$ at which we find more peaks which deters the transfer of heat. In the range $2.25 \leq P \leq 7$ particles move very slowly taking infinitesimal displacement which make the convection faster for any fluid having this property.



Figure 3 Convection as the relation between the Nusselt numbers and Rayleigh numbers with difference of Chandrasekhar numbers Q at Prandtl number P = 100

Figure 3 shows the change of magnetic field, which increases gradually reaching Q = 1000 with high Prandtl number P = 100. In this case we get strong magnetic flux. In the stress-free problem the Nusselt number takes high value Nu = 10000000, which appears at Sun's corona. The figure shows peaks below zero degree Kelvin which is mathematically acceptable but physically unacceptable, this discrepancy

stems from neglecting the equation of state in driving the used equations from solution.

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El-Kholy, SA

Appendix

$$\begin{split} \varphi \ (z) &= \frac{1}{128 \left(-1 + E^{\frac{s}{p}}\right) p^2 Q^3} \left(1 - E^{\frac{s}{p}} + \right. \\ &= \frac{1 + s \left(3 - 2z\right) - 2z}{4p} - E^{-\frac{\left(1 + s\right) \left(-1 + 2z\right)}{4p}} - E^{\frac{\left(-1 + s\right) \left(1 + 2z\right)}{4p}} + E^{\frac{1 - 2z + s \left(3 + 2z\right)}{4p}} + S \left(E^{\frac{-2 + s}{2p}} - E^{\frac{1 + s}{2p}}\right) - E^{\frac{-1 + s \left(3 - 2z\right) - 2z}{4p}} + \\ &= \frac{1 - 2z + s \left(3 + 2z\right)}{4p} - E^{\frac{\left(-1 + s\right) \left(1 + 2z\right)}{4p}} + E^{\frac{1 - 2z + s \left(3 + 2z\right)}{4p}} + \\ &= 2 \left(-1 - E^{\frac{s}{p}} + E^{\frac{-1 + s}{2p}} \left(1 + E^{\frac{1}{p}}\right)\right) z\right) + \\ &4 p^2 Q \left(2 \left(1 - E^{\frac{s}{p}} + E^{\frac{-1 + s \left(3 - 2z\right) - 2z}}{4p} - E^{-\frac{\left(1 + s\right) \left(1 - 2z\right)}{4p}} - \\ &= E^{\frac{\left(-1 + s\right) \left(1 + 2z\right)}{4p}} + E^{\frac{1 - 2z + s \left(3 + 2z\right)}{4p}}\right) - \\ &\left(-1 + E^{\frac{s}{p}}\right) Q \left(-1 + 4 z^2\right)\right) \right) \end{aligned} \\ \theta \ (z) &= \left(E^{-\frac{\left(1 + s\right) \left(-1 + 2z\right)}{4p}} \left(1 + 8 p^2 Q + p \left(-1 + s\right) - s\right) + \\ &\left(1 - \frac{\left(-1 + E^{\frac{-1 + s}{2p}}\right) \left(1 + 8 p^2 Q + s - p \left(1 + s\right)\right)}{1 + s}\right) \right) \\ \left(16 \left(-1 + E^{\frac{s}{p}}\right) Q^2 \left(1 - p + 4 p Q\right)\right) - \\ &\left(E^{\frac{-1 + 2 - 2 - z}{2p}} \left(64 \left(-E^{\frac{1}{2p}} + E^{\frac{2 - s}{2p}} + E^{\frac{2 - 2 - s}{2p}} - E^{\frac{1 + 2 - s}{2p}}\right) p^4 Q^2 s - \\ &\left(-E^{\frac{1}{2p}} + E^{\frac{s}{2p}} + E^{\frac{2 + s}{2p}} - E^{\frac{1 + 2 - s}{2p}}\right) \\ &p \left(-1 + 4 Q\right) s \left(-1 + s^2\right) + \end{aligned}$$

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$$\begin{split} & 16 p^2 Q \left(\left(-E^{\frac{1}{2p}} + E^{\frac{s}{2p}} + E^{\frac{2+s}{2p}} - E^{\frac{1+2s}{2p}} \right) s - \\ & E^{\frac{1}{2p}} \left(-1 + E^{\frac{s}{p}} \right) Q \left(-1 + s^2 \right) - \\ & 4 E^{\frac{1}{2p}} \left(-1 + E^{\frac{s}{p}} \right) Q^2 R \left(-1 + s^2 \right) \right) - \\ & 16 p^3 Q \left(\left(-E^{\frac{1}{2p}} + E^{\frac{s}{2p}} + E^{\frac{2+s}{2p}} - E^{\frac{1+2s}{2p}} \right) s - \\ & E^{\frac{1}{4r}} \left(-1 + E^{\frac{1}{r}} \right) Q \left(-1 + E^{\frac{1}{r}} \right) Q^2 \left(-1 + R \right) \left(-1 + s^2 \right) + \\ & 16 E^{\frac{1}{2r}} \left(-1 + E^{\frac{1}{r}} \right) Q^3 R \left(-1 + s^2 \right) \right) \right) \right) \right) \\ & \left(64 \left(-1 + E \right) \left(-1 + E^{\frac{s}{p}} \right) \\ & p^2 Q^3 \left(1 - p + 4 p Q \right) \left(-1 + s \right) \left(1 + s \right) \right) - \\ & \frac{1}{1 - E} \left(\sqrt{E} \left(-\sqrt{E} \left(-\frac{1}{32Q} + \left(16 \left(-1 + E^{\frac{s}{p}} \right) \right) \right) \right) \right) \\ & \left(128 \left(-1 + E^{\frac{s}{p}} \right) p^2 Q^3 \right) \cdot \left(-1 + E^{\frac{s}{p}} + E^{\frac{1+s}{2}} \right) s \right) / \\ & \left(128 \left(-1 + E^{\frac{s}{p}} \right) p^2 Q^3 \right) \cdot \left(-1 + E^{\frac{s}{p}} \right) s + p \left(1 - E^{\frac{s}{p}} \right) \\ & \left(-2 + 4 E^{\frac{-1+s}{2p}} + 2 E^{\frac{s}{p}} \right) s + \left(-1 + E^{\frac{s}{p}} \right) s \right) + p \left(1 - E^{\frac{s}{p}} \right) \\ & \left(-2 + 4 E^{\frac{-1+s}{2p}} - 2 E^{\frac{s}{p}} \right) s - \left(-1 + E^{\frac{s}{p}} \right) s^2 \right) \right) / \\ & \left(16 \left(-1 + E^{\frac{s}{p}} \right) \\ & Q^2 \left(1 - p + 4 p Q \right) \left(-1 + s \right) \left(1 + s \right) \right) \right) + \\ & \frac{1}{\sqrt{E}} \left(-\frac{1}{32Q} + \left(-16 \left(-1 + E^{\frac{s}{p}} \right) \right) \\ & P^2 Q^2 \left(1 + 4 Q R \right) + \left(1 - E^{\frac{-1+s}{2p}} + E^{\frac{s}{p}} - E^{\frac{1+s}{2p}} \right) s \right) / \\ & \left(128 \left(-1 + E^{\frac{s}{p}} \right) p^2 Q^3 \right) + \left(-1 + E^{\frac{s}{p}} \right) \right) \right) + \\ \end{array}$$

Q

$$2 \left(1 + E^{\frac{s}{p}} - 2 E^{\frac{1+s}{2p}}\right) s + \left(-1 + E^{\frac{s}{p}}\right) s^{2} - 8 p^{2} Q$$

$$\left(1 - E^{\frac{s}{p}} + \left(1 + E^{\frac{s}{p}} - 2 E^{\frac{1+s}{2p}}\right) s\right) + p \left(1 - E^{\frac{s}{p}} + 2 \left(1 + E^{\frac{s}{p}} - 2 E^{\frac{1+s}{2p}}\right) s - \left(-1 + E^{\frac{s}{p}}\right) s^{2}\right)\right) / (16 \left(-1 + E^{\frac{s}{p}}\right) Q^{2}$$

$$\left(1 - p + 4 p Q\right) \left(-1 + s\right) \left(1 + s\right) \left(1 + s\right)\right) \right) \right) + (\left(16 \left(-1 + E^{\frac{s}{p}}\right) p^{2} Q^{2} \left(1 + 4 Q R\right) + \left(-1 + E^{\frac{-1+s}{2p}} - E^{\frac{s}{p}} + E^{\frac{1+s}{2p}}\right) s\right) z\right) / (64 \left(-1 + E^{\frac{s}{p}}\right) p^{2} Q^{3}) - \frac{z^{2}}{8 Q}$$

Where $s = \sqrt{1 + 16p^2Q}$.