# Effect of temperature dependent elastic property of materials on wave propagation in microstretch generalized thermoelastic solid 

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#### Abstract

The present paper is aimed at studying the effect of temperature dependent properties of elastic materials on wave propagation in a microstretch generalized thermoelastic solid. The mathematical model has been simplified by using the Helmholtz decomposition technique and secular equations connecting phase velocity with wave number, for symmetric and skew symmetric wave modes are derived. Phase velocity, attenuation coefficients, amplitude ratios and specific loss are obtained. The results obtained are compared with those obtained by author previous work. Finally, in order to illustrate the analytical developments, the numerical solution of secular equations, amplitude ratios and specific loss with wave number for different angle of inclination is carried out for magnesium crystal material with the help of Descartes’ algorithm. This type of study has many applications in various fields of science and technology.


## Introduction

The concept of microcontinuum, proposed by Eringen (1999), can take into account the microstructure effects while the theory itself is still a continuum formulation. The first grade microcontinuum consists a hierarchy of theories, namely, micropolar, microstretch and micromorphic, depending on how much microdegrees of freedom are incorporated. These high order continuum theories are considered to be potential tools to model the behavior of the material with a complicated microstucture. For example, in the case of a foam composite, when the size of the reinforced phase is comparable to the intrinsic length scale of the foam. In these situations, the microstucture of the foam must be taken into account to some degree, so a high order continuum model must be assigned for the foam matrix. The same remains true for nanocomposites, since the scale of the reinforced phase is so small, the surroundings matrix cannot be homogenized as a simple material (Cauchy medium), some intrinsic microstructures of the matrix must be considered in a proper continuum model.

Microstretch theory is a generalization of the theory of micropolar elasticity and a special case of the micromorphic theory. The microstrech solids are those in which material particles can undergo stretches (expansion and contraction) in addition to translation and rotation. Thus a microstretch elastic solid possesses seven degrees of freedom: three for translation, three for rotation (as in micropolar elasticity) and one for stretch, required by substructures. Such a generalized media can catch more detailed information about the microdeformation inside a material point, which is more suitable for modeling the overall property of the foam matrix in the case of foam composites.

Kumar and Singh (1998) studied wave propagation in a generalized thermomicrostretch elastic solid. De Cicco (2003) investigated the stress concentration effects in microstretch elastic bodies. Liu and Hu (2004) investigated the inclusion problem of microstretch. Svanadze (2004) constructed fundamental solution of the system of equations of steady oscillations in the theory of microstretch elastic solids. Bakshi et. al. (2007) investigated the problem of wave propagation in materials with memory in generalized thermoelasticity. Recently Kumar et. al. (2007) discussed various problems of wave propagation in microstretch elastic medium.

The elastic modulus is an important physical property of materials reflecting the elastic deformation capacity of the material when subjected to an applied external load. Most of the investigations were done under the assumption of the temperature-independent material properties, which limit the applicability of the solutions obtained to certain ranges of temperature. Modern structural elements are often subjected to temperature change of such magnitude that their material properties may no longer be regarded as having constant values even in an approximate sense.

At high temperature the material characteristics such as the modulus of elasticity, coefficient of thermal expansion and thermal conductivity etc. are no longer constants. The thermal and mechanical properties of the materials vary with temperature, so the temperature-dependence of the material properties must be taken into consideration in the thermal stress analysis of these elements. Tanigawa (1995) investigated thermoelastic problems for non-homogeneous structural material. Ezzat et al $(2001,2004)$ investigated the dependence of modulus of elasticity on reference temperature in generalized thermoelasticity and obtained interesting results.

The problem of wave propagation in generalized microstretch thermoelastic medium in which elastic constants are taken as linear function of temperature has been analyzed in this paper. The physical applications are encountered in the context of problems such as ground explosions and oil industries. This problem is useful in the field of earthquake engineering and geomechanics, where interest is in various phenomenons occurring in earthquakes and measurements of stresses and temperature distribution.

## Basic Equations

The basic equations in a linear homogeneous, isotropic microstretch generalized thermoelastic solid in the absence of body forces, body couples, stretch force and heat sources are given by:

## Balanced Laws

## Balance of Momentum

$$
\begin{equation*}
t_{k l, k}=\rho \ddot{u}_{l} \tag{1}
\end{equation*}
$$

## Balance of Moment of Momentum

$$
\begin{equation*}
m_{k l, l}+\varepsilon_{l m n} t_{m n}=\rho \dot{j} \ddot{\phi}_{m} \tag{2}
\end{equation*}
$$

## Balance of first stress moments

$$
\begin{equation*}
\lambda_{k, k}^{*}+\left(t_{k k}-s\right)=\rho j_{0} \ddot{\phi}^{*} \tag{3}
\end{equation*}
$$

## Energy Equation

$$
\begin{equation*}
\rho \dot{\eta} T_{0}=q_{i, i} . \tag{4}
\end{equation*}
$$

## Constitutive Relations

$t_{k l}=\lambda u_{r, r} \delta_{k l}+\mu\left(u_{k, l}+u_{l, k}\right)+K\left(u_{l, k}-\varepsilon_{k l r} \phi_{r}\right)+\lambda_{0} \delta_{k l} \phi^{*}-v\left(1+\tau_{1} \frac{\partial}{\partial t}\right) T \delta_{k l}$,
$m_{k l}=\alpha \phi_{r, r}+\beta \phi_{k, l}+\gamma \phi_{l, k}+b_{0} \varepsilon_{m l k} \phi_{, m}^{*}$,
$\lambda_{k}^{*}=a_{0} \phi_{k}^{*}+b_{0} \varepsilon_{k l m} \phi_{l, m}$,
$s-t_{k k}=-\beta_{1} T+\lambda_{1} \phi^{*}+\lambda_{0} u_{k, k}$,
$q_{k}=K \delta_{k l} T_{, k}$,
$\rho \eta T_{0}=\rho C^{*} T+v T_{0} u_{k, k}+v_{1} T_{0} \phi^{*}$
$\left.\left(1+\tau_{0} \frac{\partial}{\partial t}\right) q_{i, i}=K^{*} T_{, i i}\right\}$ for Lord and Shulman Theory (L - S Theory) $\left.\begin{array}{l}\rho \eta T_{0}=\rho C^{*}\left(\frac{\partial}{\partial t}+\tau_{0} \frac{\partial^{2}}{\partial t^{2}}\right) T+\nu T_{0} u_{k, k}+v_{1} T_{0} \phi^{*} \\ q_{1, i}=K^{*} T\end{array}\right\} \quad$ for Green and Lindsay Theory (G - L Theory). $q_{i, i}=K^{*} T_{, i i}$
$u_{i}$ are the components of displacement vector, $\phi_{i}$ are the components of microrotation vector, $t_{i j}$ are the components of force stress tensor, $m_{i j}$ are the components of couple stress tensor, $e_{i j}$ are the components of micropolar strain tensor, satisfying $e_{i j}=u_{j, i}+\varepsilon_{j i 3} \phi_{3}$, $\lambda, \mu, K, \alpha, \beta, \gamma, \alpha_{0}, \lambda_{0}, \lambda_{1}$ are material constants, $\rho$ is the density, $j$ is the microinertia, $j_{0}$ is the
microinertia of microelements, $\phi^{*}$ is scalar point microstretch function, $\lambda_{k}$ is the component of microstress tensor, $\varepsilon_{i j k}$ is the permutation symbol, $K^{*}$ is the coefficient of thermal conductivity, $C^{*}$ is the specific heat at constant strain, $T$ is the temperature change, $T_{0}$ is the uniform temperature, $v=(3 \lambda+2 \mu+K) \alpha_{t_{1}}, v_{1}=(3 \lambda+2 \mu+K) \alpha_{t_{2}}$, where $\alpha_{t_{1}}, \alpha_{t_{2}}$ are the coefficients of linear thermal expansions. The comma notation denotes spatial derivatives. For Lord and Shulman Theory (L-S Theory) $\tau_{1}=0, n_{0}=1$. For Green and Lindsay Theory (G-L Theory) $\tau_{1} \geq \tau_{0}>0, n_{0}=0$. Also, for the Coupled theory of thermoelasticity (C-T Theory) $\tau_{1}=\tau_{0}=0$. Substituting the value from equations (5)-(10) in equations (1)-(4), we obtain

$$
\begin{align*}
& (\lambda+2 \mu+K) \nabla(\nabla \cdot \vec{U})-(\mu+K) \nabla \times \nabla \times \vec{U}+K \nabla \times \vec{\phi}+\lambda_{0} \nabla \phi^{*}-v\left(1+\tau_{1} \frac{\partial}{\partial t}\right) \nabla T=\rho \frac{\partial^{2} \vec{U}}{\partial t^{2}},  \tag{11}\\
& (\alpha+\beta+\gamma) \nabla(\nabla \cdot \vec{\phi})-\gamma \nabla \times(\nabla \times \vec{\phi})+K \nabla \times \vec{U}-2 K \vec{\phi}=\rho j \frac{\partial^{2} \vec{\phi}}{\partial t^{2}},  \tag{12}\\
& \alpha_{0} \nabla^{2} \phi^{*}+v_{1}\left(1+\tau_{1} \frac{\partial}{\partial t}\right) T-\lambda_{1} \phi^{*}-\lambda_{0} \nabla \cdot \vec{U}=\rho j_{0} \frac{\partial^{2} \phi^{*}}{\partial t^{2}},  \tag{13}\\
& K^{*} \nabla^{2} T=\rho C^{*}\left(\frac{\partial}{\partial t}+\tau_{0} \frac{\partial^{2}}{\partial t^{2}}\right)+v T_{0}\left(\frac{\partial}{\partial t}+n_{0} \tau_{0} \frac{\partial^{2}}{\partial t^{2}}\right) \nabla \cdot \vec{U}+v_{1} T_{0}\left(\frac{\partial}{\partial t}+n_{0} \tau_{0} \frac{\partial^{2}}{\partial t^{2}}\right) \phi^{*} . \tag{14}
\end{align*}
$$

## Formulation of the Problem

We considered a microstretch generalized thermoelastic medium with temperature dependent elastic modulii. The rectangular cartesian coordinate system ( $\mathrm{x}, \mathrm{y}, \mathrm{z}$ ) with z axis pointing vertically into the medium is introduced. For the two dimensional problem, we assume the components of the displacement $\vec{u}$ and microrotation vector $\vec{\phi}$, of the form

$$
\begin{equation*}
\vec{u}=(u, 0, w), \quad \vec{\phi}=\left(0, \phi_{2}, 0\right) . \tag{15}
\end{equation*}
$$

Assuming temperature dependent elastic constants of the form

$$
\begin{equation*}
\left(\lambda, \mu, \beta, K, v, v_{1}, b_{0}, \gamma, K^{*}, \alpha, \lambda_{0}\right)=\left(\lambda_{a}, \mu_{a}, \beta_{a}, K_{a}, v_{a}, v_{a 1}, b_{a 0}, \gamma_{a}, K_{a}^{*}, \alpha_{a}, \lambda_{a 0}\right)\left(1-\alpha^{*} T_{0}\right) \tag{16}
\end{equation*}
$$

where, $\alpha^{*}$ is called the empirical material constant. Also we define the dimensionless variables by the expressions:

$$
\begin{aligned}
& \left(x^{\prime}, z^{\prime}\right)=\frac{\omega^{*}}{c_{1}}(x, z), \quad\left(u^{\prime}, w^{\prime}\right)=\frac{\rho c_{1} \omega^{*}}{v_{a} T_{0}}(u, w),\left(\phi_{2}{ }^{\prime}, \phi^{*}\right)=\frac{\rho c_{1}^{2}}{v_{a} T_{0}}\left(\phi_{2}, \phi^{*}\right), \quad t_{i j}^{\prime}=\frac{t_{i j}}{v_{a} T_{0}}, \\
& \left(m_{23}^{\prime}, \lambda_{i}^{\prime}\right)=\frac{\omega^{*}}{c_{1} v_{a} T_{0}}\left(m_{23}, \lambda_{i}\right), T^{\prime}=\frac{T}{T_{0}}, t^{\prime}=\omega^{*} t,\left(\tau_{0}^{\prime}, \tau_{1}^{\prime}\right)=\omega^{*}\left(\tau_{0}, \tau_{1}\right), \omega^{\prime}=\frac{\omega}{\omega^{*}},
\end{aligned}
$$

$$
\begin{equation*}
\omega^{*}=\frac{\rho C^{*} c_{1}^{2}}{K_{a}^{*}}, \quad \rho c_{1}^{2}=\lambda_{a}+2 \mu_{a}+K_{a} . \tag{17}
\end{equation*}
$$

where $\omega^{*}$ is the characteristic frequency of the material and $c_{1}$ is the longitudinal wave velocity of the medium.
The displacement components, $u(x, z, t)$ and $w(x, z, t)$, may be written in terms of the potential
functions, $q(x, z, t)$ and $\psi(x, z, t)$, as
$u=\frac{\partial q}{\partial x}+\frac{\partial \psi}{\partial z}, \quad w=\frac{\partial q}{\partial z}-\frac{\partial \psi}{\partial x}$.

Using equations (15) - (18), equations (11) - (14) recast into the following form (after suppressing the primes):
$\left(\nabla^{2}-R \frac{\partial^{2}}{\partial t^{2}}\right) q+p_{0} \phi^{*}-\left(1+\tau_{1} \frac{\partial}{\partial t}\right) T=0$,
$\nabla^{2} \psi-\frac{p \phi_{2}}{\delta^{2}}-\frac{R}{\delta^{2}} \frac{\partial^{2} \psi}{\partial t^{2}}=0$,
$\nabla^{2} \phi_{2}+\delta^{* 2} \nabla^{2} \psi-2 \delta^{* 2} \phi_{2}=\frac{R}{\delta_{1}^{2}} \frac{\partial^{2} \phi_{2}}{\partial t^{2}}$,
$\nabla^{2} T-R\left(\frac{\partial}{\partial t}+\tau_{0} \frac{\partial^{2}}{\partial t^{2}}\right) T=\varepsilon_{1}\left(\frac{\partial}{\partial t}+n_{0} \tau_{0} \frac{\partial^{2}}{\partial t^{2}}\right) \phi^{*}+\varepsilon_{2}\left(\frac{\partial}{\partial t}+n_{0} \tau_{0} \frac{\partial^{2}}{\partial t^{2}}\right) \nabla^{2} q$,
$\nabla^{2} \phi^{*}-p_{1} \delta_{1}^{*} \phi^{*}-p_{0} \delta_{1}^{*} \nabla^{2} q+\bar{v} \delta_{1}^{*}\left(1+\tau_{1} \frac{\partial}{\partial t}\right) T=\frac{R}{\delta_{2}^{2}} \frac{\partial^{2} \phi^{*}}{\partial t^{2}}$.
where $R=\frac{1}{\left(1-\alpha^{*} T\right)}$.

## Boundary Conditions

We consider following boundary conditions at the surface of the plate $z= \pm H$.
$t_{33}=0, \quad t_{31}=0, \quad m_{32}=0, \quad \lambda_{, z}=0, \quad \frac{\partial T}{\partial z}+h T=0$,
where $h$ is the surface heat transfer coefficient:
$h \rightarrow 0$ corresponds to thermally insulated boundaries and
$h \rightarrow \infty$ refers to isothermal boundaries.

## Normal Mode Analysis and Solution of the problem

The solution of the considered physical variables can be decomposed in terms of normal modes as in the following form:
$\left(q, \psi, \phi_{2}, T, \phi^{*}\right)=\left(\bar{q}, \bar{\psi}, \bar{\phi}_{2}, \bar{T}, \bar{\phi}^{*}\right) e^{i \xi(x \sin \theta+m z-c t)}$,
where $\xi$ is the wave number, $\omega=\xi_{c}$ is the angular frequency and c is the phase velocity of the wave, $m$ is the unknown parameter which signifies the penetration depth of the wave.

Using (25) in equations (19)-(23), we obtain

$$
\begin{align*}
& \left\{\left(R \omega^{2}-\xi^{2}\left(\sin ^{2} \theta+m^{2}\right)\right) \bar{q}+p_{0} \bar{\phi}^{*}+i \omega t_{1} \bar{T}\right\} e^{i \xi(x \sin \theta-c t+m z)}=0  \tag{26}\\
& \left\{\left(\frac{R \omega^{2}}{\delta^{2}}-\xi^{2}\left(\sin ^{2} \theta+m^{2}\right)\right) \bar{\psi}-\frac{p}{\delta^{2}} \bar{\phi}_{2}\right\} e^{i \xi(x \sin \theta-c t+m z)}=0  \tag{27}\\
& \left\{-\xi^{2} \delta^{* 2}\left(m^{2}+\sin ^{2} \theta\right) \bar{\psi}+\left(\frac{R \omega^{2}}{\delta_{1}^{2}}-\xi^{2}\left(m^{2}+\sin ^{2} \theta\right)-2 \delta^{* 2}\right) \bar{\phi}_{2}\right\} e^{i \xi(x \sin \theta-c t+m z)}=0, \tag{28}
\end{align*}
$$

$$
\begin{align*}
& \left\{P_{0} \delta_{1}^{*} \xi^{2}\left(m^{2}+\sin ^{2} \theta\right) \bar{q}+\left(\frac{R \omega^{2}}{\delta_{2}^{2}}-\xi^{2}\left(m^{2}+\sin ^{2} \theta\right)-P_{1} \delta_{1}^{*}\right) \bar{\phi}^{*}-\bar{v} \delta_{1}^{*} i \omega t_{1} \bar{T}\right\} e^{i \xi(x \sin \theta-c t+m z)}=0  \tag{29}\\
& \left\{-\xi^{2} \varepsilon_{2} \omega^{2}\left(m^{2}+\sin ^{2} \theta\right) t_{2} \bar{q}+\omega^{2} \varepsilon_{1} t_{2} \bar{\phi}^{*}+\left(R \omega^{2} t_{0}-\xi^{2}\left(m^{2}+\sin ^{2} \theta\right)\right) \bar{T}\right\} e^{i \xi(x \sin \theta-c t+m z)}=0 \tag{30}
\end{align*}
$$

The system of equations (26)-(30) has a non-trivial solution if the determinant of coefficients of ( $\bar{q}, \bar{\phi}^{*}, \bar{T}$ ) and ( $\bar{\psi}, \bar{\phi}_{2}$ ) vanishes, which yields an algebraic equations relating m to c.

So we have
$\left(\xi^{6} m^{6}+H \xi^{4} m^{4}+I \xi^{2} m^{2}+J\right)\left(\bar{q}, \bar{\phi}^{*}, \bar{T}\right)=0$,
and

$$
\begin{equation*}
\left(\xi^{4} m^{4}+F \xi^{2} m^{2}+G\right)\left(\bar{\psi}, \bar{\phi}_{2}\right)=0 \tag{32}
\end{equation*}
$$

where

$$
\begin{aligned}
& A=-R \omega^{2}\left(\frac{1}{\delta_{1}^{2}}-\frac{1}{\delta^{2}}\right)+\delta^{* 2}\left(\frac{2}{p}-\frac{p}{\delta^{2}}\right), \quad B=\frac{R \omega^{2}}{\delta^{2}}\left(\frac{R \omega^{2}}{\delta_{1}^{2}}-2 \delta^{* 2}\right), C=\frac{R \omega^{2}}{\delta_{2}^{2}}+R \omega^{2}\left(t_{0}+1\right)+\delta_{1}^{*}\left(p_{0}^{2}-p_{1}\right) \\
& -i \omega^{3} \varepsilon_{2} t_{1} t_{2}, \quad D=-\frac{R^{2} \omega^{4}}{\delta_{2}^{2}}\left(R t_{0}+R+i \omega \varepsilon_{2} t_{1} t_{2}\right)+R \omega^{2}\left(-R \omega^{2} t_{0}+p_{1} \delta_{1}^{*}\left(t_{0}+1\right)-\omega t_{0} p_{0}^{2} \delta_{1}^{*}\right)+i \omega^{3} t_{1} t_{2} \delta_{1}^{*} \\
& \left(p_{0} \varepsilon_{1}-p_{1} \varepsilon_{1}\right)+\bar{v} \delta_{1}^{*} i \omega^{3} t_{1} t_{2}\left(\varepsilon_{2} p_{0}-\varepsilon_{1}\right), \quad E=\frac{R^{3} \omega^{6} t_{0}}{\delta_{2}^{2}}-R^{2} \omega^{4} t_{0} p_{1} \delta_{1}^{*}+R \bar{v} i \omega^{5} \delta_{1}^{*} \varepsilon_{1} t_{1} t_{2}, \\
& F=2 \xi^{2} \sin ^{2} \theta+A, \quad G=\xi^{4} \sin ^{4} \theta+A \xi^{2} \sin ^{2} \theta+B, H=3 \xi^{2} \sin ^{2} \theta+C, \\
& I=3 \xi^{4} \sin ^{4} \theta+2 \xi^{2} \sin ^{2} \theta C+D, \quad J=\xi^{6} \sin ^{6} \theta+C \xi^{4} \sin ^{4} \theta+D \xi^{2} \sin ^{2} \theta+E .
\end{aligned}
$$

Equation (31) is cubic in $m$, and three values correspond to three value of c. These roots give the analytical expressions for the velocities of propagation of three types of waves, namely longitudinal displacement wave (LD), longitudinal microstretch wave (LMS) and thermal wave (T-Wave). Similarly, equation (32) is quadratic equation which gives two values of $m$, and hence two values of c. This equation is coupled in $\bar{\psi}, \bar{\phi}_{2}$ which correspond to coupled transverse displacement and microrotational waves (CD-I, CD-II).

Equations (31) and (32) lead to the following solution for potential functions, microrotation, microstretch and temperature as:
$q=\sum_{i=1}^{3}\left(A_{i} \cos m_{i} \xi z+B_{i} \sin m_{i} \xi z\right) e^{i \xi(x \sin \theta-c t)}$,
$\psi=\sum_{i=4}^{5}\left(A_{i} \cos m_{i} \xi z+B_{i} \sin m_{i} \xi z\right) e^{i \xi(x \sin \theta-c t)}$,
$\phi_{2}=\frac{\delta^{2}}{p} \sum_{j=4}^{5} f_{j}\left(A_{j} \cos m_{j} \xi Z+B_{j} \sin m_{j} \xi z\right) e^{i \xi(x \sin \theta-c t)}$,
$\phi^{*}=\sum_{i=1}^{3}\left(A_{i} \cos m_{i} \xi_{z}+B_{i} \sin m_{i} \xi z\right) V_{i} e^{i \xi(x \sin \theta-c t)}$,
$T=\sum_{i=1}^{3}\left(A_{i} \cos m_{i} \xi_{z}+B_{i} \sin m_{i} \xi_{z}\right) S_{i} e^{i \xi(x \sin \theta-c t)}$,
where

$$
\begin{aligned}
& V_{i}=\frac{-\delta_{1}^{*}\left[\left(\bar{v}-p_{0}\right) m_{i}^{2} \xi^{2}-\bar{v} \xi^{2}\left(R c^{2}+p_{0} / \bar{v}-1\right)\right]}{m_{i}^{2} \xi^{2}-\xi^{2}\left[c^{2}\left(R / \delta_{2}^{2}-\delta_{1}^{*} / \omega^{2}\left(\bar{v} p_{0}-p_{1}\right)-\sin ^{2} \theta\right)\right]}, \\
& S_{i}=\frac{\left\{m_{i}^{2} \xi^{2}-R \xi^{2}\left(c^{2}-\sin ^{2} \theta\right)\left[m_{i}^{2} \xi^{2}-\xi^{2}\left(c^{2}\left(\frac{R}{\delta_{2}^{2}}-\frac{\delta_{1}^{*}}{\omega^{2}}\left(\bar{v} p_{0}-p_{1}\right)\right)-\sin ^{2} \theta\right)\right]+\delta_{1}^{*} p_{0}\left[\left(\bar{v}-p_{0}\right) m_{i}^{2} \xi^{2}-\bar{v} \xi^{2}\left(R c^{2}+p_{0} / \bar{v}-\sin ^{2 \theta}\right)\right]\right\}}{i \omega t_{1}\left[m_{i}^{2} \xi^{2}-R \xi^{2}\left(c^{2}-\sin ^{2} \theta\right)\left[m_{i}^{2} \xi^{2}-\xi^{2}\left(c^{2}\left(\frac{R}{\delta_{2}^{2}}-\frac{\delta_{1}^{*}}{\omega^{2}}\left(\bar{v} p_{0}-p_{1}\right)\right)-\sin ^{2} \theta\right)\right]\right.} .
\end{aligned}
$$

With the help of equations (33) and (34) in equation (18) we obtain the displacement components as:

$$
\begin{aligned}
u_{1}= & \left(i \xi \sin \theta\left(A_{1} \cos m_{1} \xi z+A_{2} \cos m_{2} \xi z+A_{3} \cos m_{3} \xi z+B_{1} \sin m_{1} \xi z+B_{2} \sin m_{2} \xi z+B_{3} \sin m_{3} \xi z\right)\right. \\
& \left.+m_{4} \xi\left(B_{4} \cos m_{4} \xi z-A_{4} \sin m_{4} \xi z\right)+m_{5} \xi\left(B_{5} \cos m_{5} \xi z-A_{5} \sin m_{5} \xi z\right)\right) e^{i \xi(x \sin \theta-c t)}, \\
u_{2}= & \left(m_{1} \xi\left(-A_{1} \sin m_{1} \xi z+B_{1} \cos m_{1} \xi z\right)+m_{2} \xi\left(-A_{2} \sin m_{2} \xi z+B_{2} \cos m_{1} \xi z\right)+m_{3} \xi\left(-A_{3} \sin m_{3} \xi z+B_{3} \cos m_{3} \xi z\right)\right. \\
& \left.-i \xi \sin \theta\left(A_{4} \cos m_{4} \xi z+B_{4} \sin m_{4} \xi z+A_{5} \cos m_{5} \xi z+B_{5} \sin m_{5} \xi z\right)\right) e^{i \xi(x \sin \theta-c t)}
\end{aligned}
$$

## Derivation of the secular equations

Using the boundary condition (24) on the surfaces $z= \pm H$ of the plate and with the help of equations (33)-(37), we obtain a system of ten secular equations:

$$
\begin{align*}
& \sum_{i=1}^{3} P\left(A_{i} c_{i}+B_{i} s_{i}\right)+\sum_{j=4}^{5} Q\left(-A_{j} s_{j} m_{j}+B_{j} c_{j} m_{j}\right)=0  \tag{38}\\
& \sum_{i=1}^{3} P\left(A_{i} c_{i}-B_{i} s_{i}\right)+\sum_{j=4}^{5} Q\left(A_{j} s_{j} m_{j}+B_{j} c_{j} m_{j}\right)=0  \tag{39}\\
& \sum_{i=1}^{3} Q\left(-A_{i} m_{i} s_{i}+B_{i} m_{i} c_{i}\right)+\sum_{j=4}^{5} P\left(A_{j} c_{j}+B_{j} s_{j}\right)=0  \tag{40}\\
& \sum_{i=1}^{3} Q\left(A_{i} m_{i} s_{i}+B_{i} m_{i} c_{i}\right)+\sum_{j=4}^{5} P\left(A_{j} c_{j}-B_{j} s_{j}\right)=0  \tag{41}\\
& \sum_{i=1}^{3} R^{\prime} V_{i}\left(A_{i} c_{i}+B_{i} s_{i}\right)+\sum_{j=4}^{5} S f_{j} m_{j}\left(-A_{j} s_{j}+B_{j} c_{j}\right)=0  \tag{42}\\
& \sum_{i=1}^{3} R^{\prime} V_{i}\left(A_{i} c_{i}-B_{i} s_{i}\right)+\sum_{j=4}^{5} S f_{j} m_{j}\left(A_{j} s_{j}+B_{j} c_{j}\right)=0  \tag{43}\\
& \sum_{i=1}^{3} U V_{i} m_{i}\left(-A_{i} s_{i}+B_{i} c_{i}\right)+\sum_{j=4}^{5} V f_{j}\left(A_{j} c_{j}+B_{j} s_{j}\right)=0  \tag{44}\\
& \sum_{i=1}^{3} U V_{i} m_{i}\left(A_{i} s_{i}+B_{i} c_{i}\right)+\sum_{j=4}^{5} V f_{j}\left(A_{j} c_{j}-B_{j} s_{j}\right)=0  \tag{45}\\
& \sum_{i=1}^{3} S_{i}\left[\left(-m_{i} s_{i}+h c_{i}\right) A_{i}+\left(m_{i} c_{i}+h s_{i}\right) B_{i}\right)=0,  \tag{46}\\
& \sum_{i=1}^{3} S_{i}\left[\left(m_{i} s_{i}+h c_{i}\right) A_{i}+\left(m_{i} c_{i}-h s_{i}\right) B_{i}\right)=0 \tag{47}
\end{align*}
$$

where

$$
P=-\delta^{2}\left[\frac{R \omega^{2}}{\delta^{2}}-2 \xi^{2}+\frac{p \xi^{2}}{\delta^{2}}\right], \quad Q=i \xi^{2}\left[\frac{\lambda_{a} \sin \theta}{\rho c_{1}^{2}}-1\right], \quad S=\frac{\gamma_{a} \delta^{2}}{p}, \quad R^{\prime}=\frac{i \xi b_{0} \sin \theta}{\rho c_{1}^{2}},
$$

$$
U=\alpha_{0}, \quad V=\frac{b_{0} i \xi \delta^{2} \sin \theta}{p}, \quad f_{j}=\frac{R \omega^{2}}{\delta^{2}}-\xi^{2}\left(m_{j}^{2}+1\right), \quad s_{k}=\sin m_{k} \xi \mathrm{z}, \quad c_{k}=\cos m_{k} \xi \mathrm{z}, \mathrm{k}=1-5 .
$$

The system of equations (38)-(47) has a non-trivial solution if the determinant of the coefficients of the amplitudes vanishes. This after lengthy algebraic reduction and manipulations leads to the secular equations for the plate with force stress free and couple stress free thermally insulated boundary as

$$
\begin{aligned}
& {\left[\frac{T_{1}}{T_{4}}\right]^{ \pm}-\frac{m_{1} \mathrm{AT1}}{m_{2} A T 2}\left[\frac{T_{2}}{T_{4}}\right]^{ \pm}+\frac{m_{1} A T 3}{m_{3} A T 2}\left[\frac{T_{3}}{T_{4}}\right]^{ \pm}+\frac{R^{\prime} V\left(f_{5}-f_{4}\right) m_{1} S_{1} l_{2}}{S U\left(m_{5} f_{5} T_{4}-m_{4} f_{4} T_{5}\right) m_{2} m_{3} A T 2}\left\{\left[\frac{T_{2} T_{3}}{T_{4}^{2} T_{5}}\right]^{ \pm}+\frac{m_{2} S_{2} l_{3}}{m_{1} S_{1} l_{2}}\left[\frac{T_{1} T_{3}}{T_{4}^{2} T_{5}}\right]\right.} \\
& \left.+\frac{m_{3} S_{3} l_{1}}{m_{1} S_{1} l_{2}}\left[\frac{T_{1} T_{2}}{T_{4}^{2} T_{5}}\right]^{ \pm}\right\}+\frac{Q}{P} \frac{\left(m_{4}-m_{5}\left(T_{5} / T_{4}\right)^{ \pm}\right)}{\left(m_{5} f_{5}-m_{4} f_{4}\left(T_{5} / T_{4}\right)^{ \pm}\right)}\left\{\begin{array}{l}
\frac{-V f_{5} f_{4} l_{5}}{U A T 2}\left[\left[\frac{T_{1}}{T_{4}}\right]^{ \pm}+\frac{m_{1} l_{6}}{m_{2} l_{5}}\left[\frac{T_{2}}{T_{4}}\right]^{ \pm}-\frac{m_{1} l_{4}}{m_{2} l_{5}}\left[\frac{T_{3}}{T_{4}}\right]^{ \pm}\right]+ \\
\frac{R}{S}\left[\left[\frac{T_{1}}{T_{4}}\right]^{ \pm}-\frac{m_{1} V_{2} A T 1}{m_{2} V_{1} A T 2}\left[\frac{T_{2}}{T_{4}}\right]^{ \pm}+\frac{m_{1} V_{3} A T 3}{m_{3} V_{1} A T 2}\left[\frac{T_{3}}{T_{4}}\right]^{ \pm}\right.
\end{array}\right\}\{\$ \\
& =\frac{Q^{2} m_{1} m_{4} m_{5}\left[l_{1} l_{5}-l_{2} l_{4}\right]\left(f_{5}-f_{4}\right)}{P^{2} \operatorname{AT2}\left(m_{5} f_{5}-m_{4} f_{4}\left[\frac{T_{5}}{T_{4}}\right]^{ \pm}\right)}+\frac{Q^{2} R V\left(m_{5} f_{4} T_{4}-m_{4} f_{5} T_{5}\right) V_{1} l_{5}}{P^{2} S U\left(m_{5} f_{5} T_{4}-m_{4} f_{4} T_{5}\right) A T 2}\left\{\left[\frac{T_{1}}{T_{4}}\right]^{ \pm}+\frac{m_{1} V_{2} l_{6}}{m_{2} V_{1} l_{5}}\left[\frac{T_{2}}{T_{4}}\right]^{ \pm}-\frac{m_{1} V_{3} l_{4}}{m_{3} l_{1} l_{5}}\left[\frac{T_{3}}{T_{4}}\right]^{ \pm}\right\}
\end{aligned}
$$

where

$$
\begin{aligned}
& A T 1=\left(V_{1} S_{3}-V_{3} S_{1}\right), \quad A T 2=\left(V_{2} S_{3}-V_{3} S_{2}\right), \quad A T 3=\left(V_{1} S_{2}-V_{2} S_{1}\right), \quad l_{1}=V_{1}-V_{2}, \\
& l_{2}=V_{2}-V_{3}, \quad l_{3}=V_{3}-V_{1}, \quad l_{4}=S_{1}-S_{2}, \quad l_{5}=S_{2}-S_{3}, \quad l_{6}=S_{3}-S_{1} .
\end{aligned}
$$

where $T_{k}=\tan m_{k} \xi d \quad \mathrm{k}=1-5$.

## Specific Loss

The specific loss, $\Delta \mathrm{W} / \mathrm{W}$ is $4 \pi$ times the absolute value of the ratio of the imaginary part of $\xi$ to the real part of

$$
\frac{\Delta \mathrm{W}}{\mathrm{~W}}=4 \pi\left|\frac{\operatorname{Im}(\xi)}{\operatorname{Re}(\xi)}\right|
$$

Kolsky (1963) notes that specific loss is the most direct method of defining internal friction for a material. Specific loss is the ratio of the energy dissipated in taking a specimen through a stress cycle, $\Delta \mathrm{W}$ to the elastic energy stored in the specimen when the strain is a maximum, W .
For a plane sinusoidal wave of small amplitude Kolsky (1963) shows that the specific loss $\Delta \mathrm{W} / \mathrm{W}$ is given by above formula.

## Amplitudes of displacement, microrotation, microstretch and temperature distribution

The amplitudes of displacement components, microrotation, microstretch and temperature distribution for symmetric and skew symmetric modes of plane waves can be obtained as:

$$
\begin{align*}
& \left(\left(u_{1}\right)_{s y},\left(u_{1}\right)_{a s y}\right)=\left(i \xi \sin \theta \sum_{i=1}^{3}\left(A_{i} \cos m_{i} \xi z, B_{i} \sin m_{i} \xi z\right)+\xi \sum_{j=4}^{5}\left(B_{j} m_{j} \cos m_{j} \xi z,-A_{j} m_{j} \sin m_{j} \xi z\right)\right) e^{i \xi(x \sin \theta-c t)} \\
& \left(\left(u_{3}\right)_{s y},\left(u_{3}\right)_{a s y}\right)=\left(\sum_{i=1}^{3} m_{i} \xi\left(-A_{i} \sin m_{i} \xi z, B_{i} \cos m_{i} \xi z\right)-i \xi \sin \theta \sum_{j=4}^{5}\left(B_{j} \sin m_{j} \xi z, A_{j} \cos m_{j} \xi z\right)\right) e^{i \xi(x \sin \theta-c t)} \\
& \left(\left(\phi_{2}\right)_{s y},\left(\phi_{2}\right)_{a s y}\right)=\left(\sum_{j=4}^{5} f_{j}\left(B_{j} \sin m_{i} \xi z, A_{j} \cos m_{i} \xi z\right) e^{i \xi(x \sin \theta-c t)},\right. \\
& \left(\left(\phi^{*}\right)_{s y},\left(\phi^{*}\right)_{a s y}\right)=\left(\sum_{j=4}^{5} f_{j}\left(B_{j} \sin m_{i} \xi z, A_{j} \cos m_{i} \xi z\right) e^{i \xi(x \sin \theta-c t)}\right. \\
& \left((T)_{s y},(T)_{a s y}\right)=\left(\sum_{i=1}^{3}\left(B_{i} s_{i} \sin m_{i} \xi z, A_{i} s_{i} \cos m_{i} \xi z\right)\right) e^{i \xi(x \sin \theta-c t)} \tag{48}
\end{align*}
$$

## Particular Cases

(i) We obtain the corresponding expressions in the case of three different theories of thermoelasticity by taking
a) $\tau_{1}=0, n_{0}=1, \quad$ for Lord and Shulman Theory (L-S theory )
b) $\tau_{1}>0, n_{0}=1, \tau_{1} \geq \tau_{0}>0, \quad$ for Green and Lindsay Theory (G-L Theory)
c) $\tau_{0}=\tau_{1}=0$, and $R=1, \quad$ for Coupled Theory (C-T Theory).
(ii) Substituting $R=1$, we obtain the corresponding expressions for microstretch thermoelastic solid.

## Numerical Results and Discussion

In order to illustrate the theoretical results obtained in the preceding sections, we now present some numerical results. The following relevant physical constants are chosen for magnesium crystal like material,

$$
\rho=1.74 \times 10^{3} \mathrm{Kg} / \mathrm{m}^{3}, \quad \lambda=9.4 \times 10^{10} \mathrm{~N} / \mathrm{m}^{2}, \quad \mu=4.0 \times 10^{10} \mathrm{~N} / \mathrm{m}^{2}, \quad K=1.0 \times 10^{10} \mathrm{~N} / \mathrm{m}^{2},
$$

$$
\gamma=0.779 \times 10^{9} \mathrm{~N}, \quad \lambda_{0}=0.85 \times 10^{10} \mathrm{~N} / \mathrm{m}^{2}, \quad \lambda_{1}=0.75 \times 10^{10} \mathrm{~N} / \mathrm{m}^{2}, \quad \alpha_{0}=1.779 \times 10^{-9} \mathrm{~N}
$$

$$
j=0.2 \times 10^{-19} \mathrm{~m}^{2}, \quad j_{0}=0.185 \times 10^{-19} \mathrm{~m}^{2}, \quad T_{0}=300^{0} \mathrm{~K}, \quad C^{*}=0.23 \times 10^{-3} \mathrm{~J} / \mathrm{Kg} \operatorname{deg},
$$

$$
K^{*}=0.6 \times 10^{-6} \mathrm{~J} / \mathrm{msec} \operatorname{deg}, \mathrm{H}=1 \mathrm{~m} .
$$

In figures 1-8, the graphical representation is given for two values of angle $\theta, \theta=30^{\circ}$ and $\theta=60^{\circ}$, for comparison between the materials with and without temperature dependent elastic constants and different modes of wave propagation. The curves with dense and sparse horizontal lines represent the variation for $\alpha^{*}=0$ (TI) and $\alpha^{*}=0.5$ (TD), respectively. Whereas, the curves with dense and sparse horizontal and vertical lines represent variation for $\theta=30^{\circ}$ and for first and second mode of wave propagation, respectively. Also, the curve with dense and sparse right slant and left slant line represents variation for $\theta=60^{\circ}$ and for first and second mode of wave propagation, respectively. In figure 9-16, he curves with dense and sparse horizontal lines represent the variations for TD, while the curves with dense and sparse vertical lines represent the variations for TI.

Figures 1 and 3 shows the variation of phase velocity with wave number for symmetric and skew symmetric mode. It is observed from the figure 1 that for initial mode of propagation and for both value of angle $\theta$, the values of phase velocity for both TD and TI initially decreases and then attains a constant value, while for second mode of propagation its value decreases with decrease in wave number. Similar behavior of phase velocity is observed from figure 3, except for initial angle of inclination where its value attains a constant value about origin. Figures 2 and 4 show the variations of attenuation coefficient with respect to wave number for symmetric and skew symmetric modes. It is observed from figure 2 that the value of attenuation coefficient at initial angle of inclination and initial mode of propagation increases with increase in wave number, while at all other angle of inclination its value sharply increases in the range $0.4 \leq$ wave number $\leq .55$ and then sharply decreases with increase in wave number. However for skew symmetric mode its value increases with increase in wave number for all modes and for all the angles.
Figures 5-6 shows the variation of specific loss with respect to wave number for symmetric and skew symmetric modes. For symmetric mode the value of specific loss start with sharp initial decrease and then oscillate with increase in wave number when $\theta=60$ and TD, while for TI at initial mode of propagation of $\theta=30$ its value initially decreases, then sharply increases in the range $0.4 \leq$ wavenumber $\leq .55$, sharply decreases within the range $0.55 \leq$ wavenumber $\leq .72$, then increases with further increase in wave number. However for skew symmetric mode, at $\theta=30$ for initial mode and for both TI and TD, its value starts with sharp initial increase and then decreases to attain a constant value. Reverse behaviour is observed in the values for second mode of propagation. At $\theta=60$ for TI, TD and both the modes its value increases with increase in wave number.
Figures 7-14 shows the variation of amplitude ratios of dilatation, microrotation, microstretch and temperature distribution with increase in thickness of the layer for symmetric and skew symmetric modes. It is observed from figure 7 that the value of amplitude ratio of dilatation for symmetric mode start with constant value about origin and then decreases with increase in thickness h when $\theta=30$, while reverse behaviour is observed in the case of $\theta=60$ and for both

TI and TD. However for symmetric mode at both angles, its value initially remains constant and then decreases with further increase in thickness of the layer, as observed from figure 8.
It is evident from figures 9 and 10 that the values of amplitude ratio of microrotation oscillate with thickness H . Figures $11-14$ show that the value of amplitude ratios of microstretch and temperature distribution for symmetric and skew symmetric modes vary in the similar way as that of dilation with difference in their magnitudes, while in the case of skew symmetric temperature distribution its value vary in opposite way.

Figures 15 and 16 shows the values of group velocity with wave number. For both symmetric and skew symmetric modes its value initially decreases and then oscillates with increase in wave number.

## Observation

Two dimensional plane wave propagation in microstretch themoelastic solid shows the existence of five type of waves namely, longitudinal displacement wave (LD), longitudinal microstretch wave (LMS) and thermal wave (T-Wave), coupled transverse displacement and microrotational waves (CD-I, CD-II) in microstretch thermoelastic solid. Due to temperature dependent properties of elastic materials, the value of phase velocity and attenuation coefficient get decreased for symmetric mode. However, for skew symmetric mode very less significant effect is observed. It is observed from above discussion that with increase in wave number the value of phase velocity and attenuation coefficient decreases for both symmetric and skew symmetric modes. Specific loss get increased with increase in wave number for symmetric mode, while for skew symmetric mode its value increases then attain a constant value with increase in wave number. Also the value of group velocity initially increases and then attains a constant value. The present theoretical and numerical analysis may be helpful to seismologists working in the field of wave propagation in solids.

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Fig. 1 Variation of Phase Velocity with wave number for Symmetric Mode


Fig. 3 Variation of Phase Velocity with wave number for Skew Symmetric mode


Fig. 2 Variation of attenuation coefficient with wave number for symmetric mode


Fig. 4 Variation of Attenuation Coefficient with wave number for Skew Symmetric mode


Fig. 5 Variations of Specific Loss with wave number for Skew Symmetric mode


Fig. 7 Variation of Group Velocity with wave number for Symmetric mode


Fig. 9 Variation of amplitude ratio of dilataion with thickness H for symmetric mode


Fig. 6 Variation of Specific Loss with wave number for Skew Symmetric mode


Fig. 8 Variation of Group Velocity with wave number for Skew Symmetric mode


Fig. 10 Variation of dilatation with thickness of layer for skew symmetric mode


Fig. 11 Variation of microrotation with thickness of layer for symmetric mode


Fig. 13 Variation of microstretch with thickness of layer H for symmetric mode


Fig. 15 Variations of temperature distribution with thickness of layer for symmetric mode


Fig. 12 Variation of microrotation with thickness of the layer H for skew symmetric mode


Fig. 14 Variation of microstretch with thickness of layer H for skew symmetric mode


Fig. 16 Variations of temperature distribution with thickness of layer for skew symmetric mode

