

## Analytical and Experimental Studies of Stability of Cylindrical Shells with a Cut-Out

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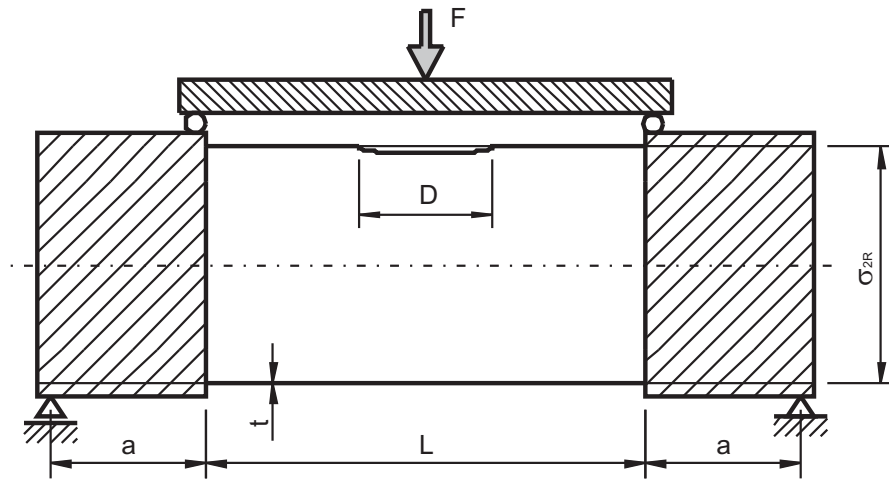
Thin-walled structure, by its nature, is subjected to loss of stability. In addition, the shell has a cutout, which reduces the critical load. The aim of this study is to determine critical stress of the shell under pure bending. The goal was achieved in two ways: experimentally and analytically. Experimental studies were carried out on a specially designed test bench with the use of a resistive strain gauge. In the analytical solution the form of buckled shell was assumed, and then Bubnov–Galerkin method was applied.

*Keywords:* Cylindrical shell, buckling, cut-out.

### 1. Introduction

Analysis of thin-walled structures must not be confined to the strength condition. Checking their stability is particularly important too. There is an extensive literature devoted to stability of the structure. An example of a monographic approach is the work of Volmir [1]. However, significant part of publications concerns ideal shells. Real structures can have many types of imperfections. The work of Staat [2] and Aghajari et al [3] are devoted to the studies on stability of cylindrical shells of variable thickness subjected to uniform external pressure. Ahn et al [4] presented comparison of the results of numerical and experimental investigation of strength of a bending pipe with local reduction of its thickness. Stability studies of an axially compressed shell with cutout or crack was described by Limam Jullien [5], Vaziri and Estekanchi [6] and Schenk and Schueller [7]. Similar studies are presented in the works of Meng–Kao Yeh and others [5] and Alashti and others [3], but in those publications a shell is subjected to bending. Wilde and others [11] presented an analytical solution of stability of cylindrical panel subjected to compression, with three edges simply supported and one edge free.

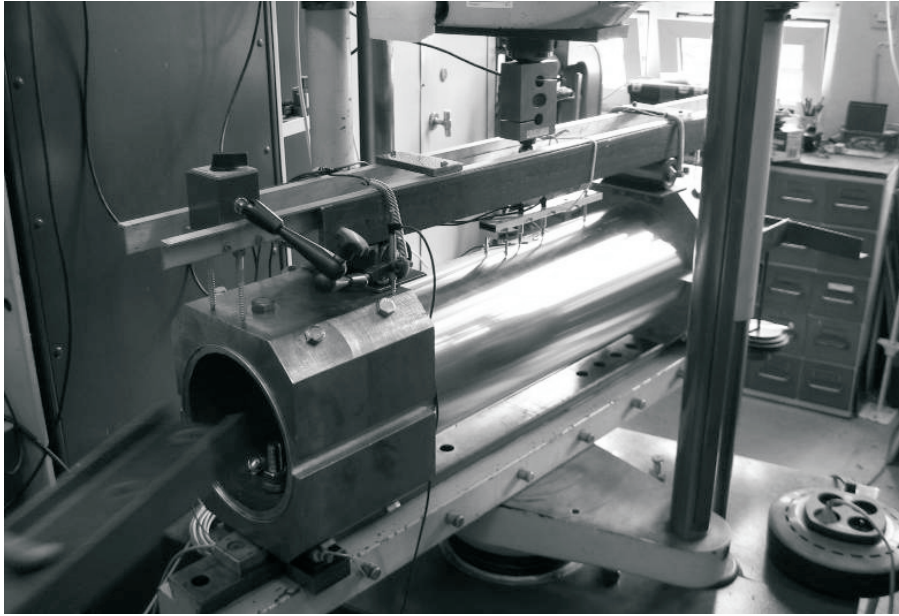
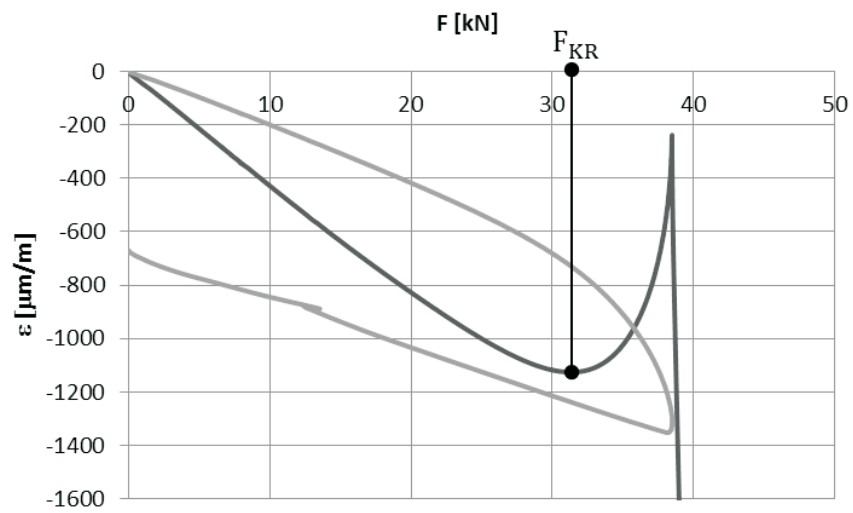
The aim of the present paper is to investigate buckling of a damaged thin-walled cylindrical shell under pure bending (Fig.1). Imperfection of the considered shell has a shape of a circular cut-out placed on the upper (compressed) generatrix of the cylinder. The paper consists of two main parts: the first one includes experimental investigation, while the second is devoted to analytical solution of the problem.



**Figure 1** Cylindrical shell with cut-out in pure bending

## 2. Experimental Investigation

A special test stand (Fig. 2) was designed and built in order to carry out experimental investigations of the buckling process in cylindrical shells in pure bending. The cylindrical shell is pivoted on its ends making use of rigid grip. The load is applied by the testing machine through the beam in a form of two forces acting on these grips. In result pure bending occurs between them, with bending moment value  $0.5Fa$ . Strain gauges are located near the cut-out. During the test the testing machine induces the load  $F$  the value of which is registered as a function of time. Similarly, indications of the strain gauges, are processed by a multichannel universal amplifier. Moreover, the rotation angle of the shell end was measured. The dimensions of the shells were as follows:  $L = 600\text{ mm}$ ,  $a = 150\text{ mm}$ ,  $R = 100\text{ mm}$ . The shells made of steel, aluminium alloy and brass were tested. The mechanical properties and thicknesses of the shells are given in Tab. 1.

**Figure 2** Test stand with the shell**Figure 3** Strain as a function of load

**Table 1** Mechanical properties and thickness of shells

Material	$R_{p0,2}$ [MPa]	$R_m$ [MPa]	A [%]	t [mm]
Steel	226.0	346.3	35.5	0.5
Aluminium alloy	128.0	128.5	0.25	0.5
Brass	289.2	393.7	31.1	0.6

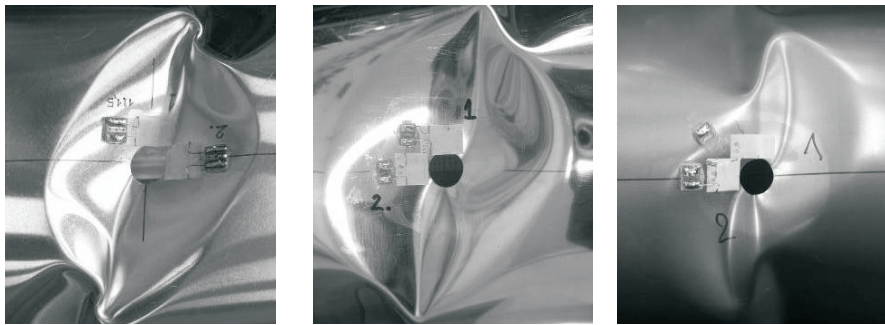
The test stand was also equipped with eight-channel strain gauge HBM Spider8 amplifier with computer data acquisition system Catman. During the study two strain gauges placed near the cut-out and three inductive sensors placed in the common base were used. An example of the relationship between the load and deformation is shown in Fig. 3. As critical load is assumed the value of force corresponding to the first extreme of strain (Fig. 3).

Symmetric and buckling modes are obtained, but antisymmetric form was dominant. Fig. 4 shows a typical form of loss of stability of the shells made of steel, brass and aluminum. These values of critical forces can be converted to the values of critical bending moments according to the equation:

$$M_{kr} = 0,5 F_{kr} a \quad (1)$$

or critical stresses according to the formula:

$$\sigma_{kr} = \frac{M_{kr}}{\pi R^2 t} = \frac{0,5 F_{kr} a}{\pi R^2 t} \quad (2)$$

**Figure 4** Antisymmetric buckling mode

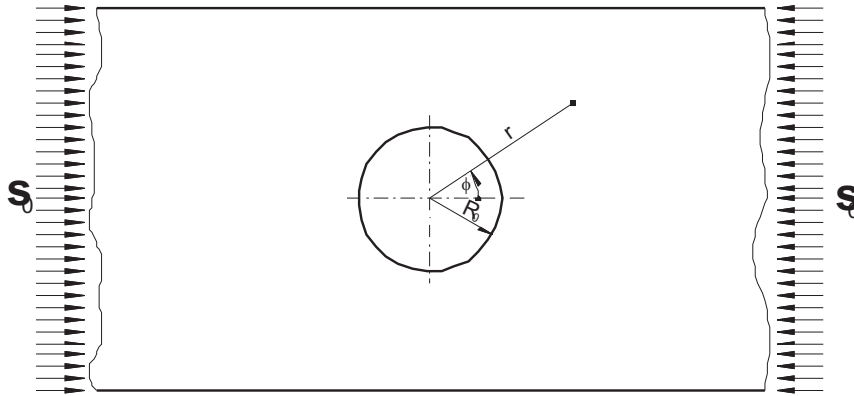
The average of three measurements (for each case) of value of critical stress is given in Tab. 2.

**Table 2** The critical stress of shell with circular cut-out

R <sub>0</sub> [mm]	Critical stress $\sigma_{kr}$ MPa		
	steel	brass	aluminium
5	183.3	203.3	96.9
14	114.6	117.4	58.3

### 3. Analytical Solution

In the analytical solution the stress distribution around a circular hole in rectangular plates (Fig. 5) is calculated based on the work of Timoshenko [11] and the equation of stability of cylindrical shells. The reason for doing so was that in the case of thin shell with a small circular hole the deviation from flatness in the surrounding area is small. Moreover, it positively affects the stiffness of given element, thus increasing the value of critical load.

**Figure 5** Rectangular plate with a circular hole

The equations describing the stress distribution around a circular hole, have a form:

$$\begin{aligned}
 \sigma_r &= \frac{1}{2} \sigma_o \left[ 1 - \frac{1}{\rho^2} + \left( 1 - \frac{4}{\rho^2} + \frac{3}{\rho^4} \right) \cos 2\varphi \right] \\
 \sigma_\varphi &= \frac{1}{2} \sigma_o \left[ 1 + \frac{1}{\rho^2} - \left( 1 + \frac{3}{\rho^4} \right) \cos 2\varphi \right] \\
 \tau_{r\varphi} &= -\frac{1}{2} \sigma_o \left[ 1 + \frac{2}{\rho^2} - \frac{3}{\rho^4} \right] \sin 2\varphi
 \end{aligned} \tag{3}$$

where:

$$\rho = \frac{r}{R_o} \quad - \text{dimensionless radius.}$$

The equation of stability of cylindrical shell:

$$D \nabla^2 \nabla^2 w + \left[ N_r \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) + N_\varphi \frac{\partial^2 w}{r^2 \partial \varphi^2} + N_{r\varphi} \frac{\partial^2 w}{r \partial r \partial \varphi} \right] = 0 \quad (4)$$

where:

$w$  – deflection of the shell,

$D$  – stiffness of the shell,

$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}$  – Laplace operator,

$N_r, N_\varphi, N_{r\varphi}$  – forces in the shell.

Deflection function is assumed in the following form:

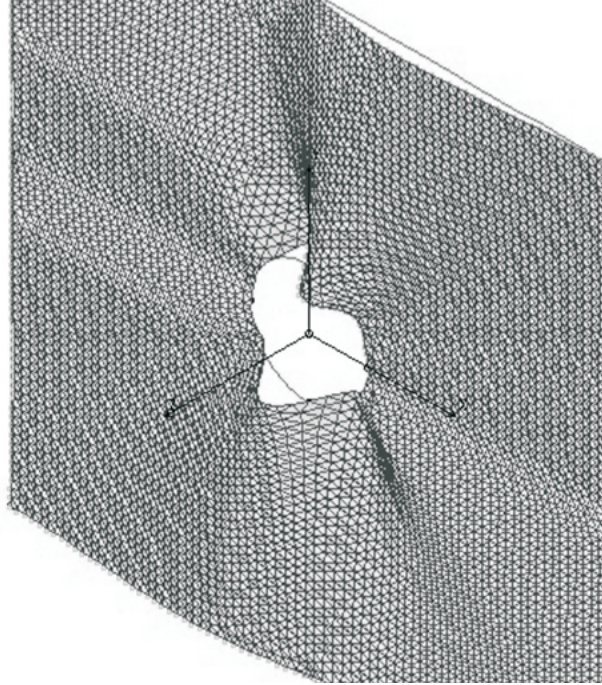
$$w(r, \varphi) = w_a \left[ \frac{4}{3\pi} \sin 2\varphi - \frac{1}{2} \sin 4\varphi + \frac{4}{5\pi} \sin 6\varphi \right] \left( \frac{R_o}{r} \right)^2 = w_a \cdot w(\varphi) \frac{1}{\rho^2} \quad (5)$$

where:

$$w(\varphi) = \frac{4}{3\pi} \sin 2\varphi - \frac{1}{2} \sin 4\varphi + \frac{4}{5\pi} \sin 6\varphi,$$

$w_a$  – constant.

Graph of the function (5) is shown in Fig.6 and corresponds to the buckling shape obtained in the experiment (Fig. 4).



**Figure 6** Graph of the deflection function near circular cut-out

Once the stress function (2) and deflection function (5) are substituted into stability equation (3), it is solved with Bubnov–Galerkin method. Orthogonal condition of equation (4) has then the form:

$$\int_1^5 \int_0^{2\pi} R(\rho, \varphi) w(\varphi) \frac{1}{\rho} d\rho d\varphi = 0 \quad (6)$$

where:

$R(\rho, \varphi)$  – left side of the equation (4),  
 $1 \leq \rho \leq 5$  – dimensionless radius.

After transformations of the condition (6) the following equation for critical load is obtained:

$$\frac{1024}{15\pi} \frac{E t^2}{12(1-\nu^2) R_o^2} = 10,662 \sigma_o \quad (7)$$

or

$$2,038 \frac{E t^2}{12(1-\nu^2) R_o^2} = \sigma_o. \quad (8)$$

The obtained value  $\sigma_o$  is a critical stress:

$$\sigma_{kr} = 2,038 \frac{E}{12(1-\nu^2)} \left( \frac{t}{R_o} \right)^2 \quad (9)$$

Besides the orthogonal condition (6) two other ones are solved with the use of the orthogonal factors:

$$w(r, \varphi) = w_a w(\varphi) \frac{1}{\rho} \quad (10)$$

$$w(r, \varphi) = w_a w(\varphi) \frac{1}{\rho^3} \quad (11)$$

They take the form:

$$\int_1^5 \int_0^{2\pi} R(\rho, \varphi) w(\varphi) d\rho d\varphi = 0 \quad (12)$$

$$\int_1^5 \int_0^{2\pi} R(\rho, \varphi) w(\varphi) \frac{1}{\rho^2} d\rho d\varphi = 0 \quad (13)$$

The solution conditions (12) and (13) gave the following expression for the critical stress:

$$\sigma_{kr} = 4,09 \frac{E}{12(1-\nu^2)} \left( \frac{t}{R_o} \right)^2 \quad (14)$$

$$\sigma_{kr} = 1,3 \frac{E}{12(1-\nu^2)} \left( \frac{t}{R_o} \right)^2 \quad (15)$$

If the deflection function is presented in the general form:

$$w(r, \varphi) = w_a w(\varphi) \frac{1}{\rho^n} \quad (16)$$

and numerical coefficients in the formulas (9), (14) and (15) are denoted by  $k$ , the general formula for the critical stress can be written as follows:

$$\sigma_{kr} = k \frac{E}{12(1 - \nu^2)} \left( \frac{t}{R_o} \right)^2 \quad (17)$$

where:

$$k = \begin{cases} 4.09 & \text{for } n = 1 \\ 2.04 & \text{for } n = 2 \\ 1.30 & \text{for } n = 3 \end{cases}$$

#### 4. Conclusions

The critical stress value (17) was determined for  $n = 1$  assuming:

- for steel:  $E = 205000$  MPa,  $\nu = 0,3$ ,
- for brass  $E = 110000$  MPa,  $\nu = 0,3$ ,
- for aluminium  $E = 69000$  MPa,  $\nu = 0,33$ .

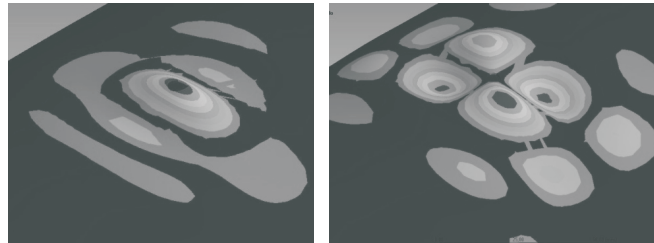
Tab. 3 summarizes the critical stress values obtained experimentally and analytically. The results are different depending on the diameter of the cut-out.

**Table 3** Critical stress of the shells. Comparison of the values obtained experimentally and analytically

$R_0$ [mm]	$\sigma_{kr}$ [MPa]					
	Steel		brass		aluminium	
	Analytical	Exper.	Analytical	Exper	Analytical	Expe.
5	751	18	602	203	258	70
14	92	115	77	117	32	58

The critical stress of shells with smaller cut-out obtained experimentally is significantly less than obtained analytically. The difference is caused by others geometrical imperfections and also by stress concentrations near rigid grip. For some specimens with cut-out of 5 mm buckling occurs away from cut-out. On the other hand in case of greater cut-out situation is different. Assumed deflection function (5) is only approximation of real behavior of the structure and depends on one free parameter only. It is necessary to expand deflection function to get better compliance of both solutions.

Numerical buckling analysis by finite element method was carried out and some results are already available. For example Fig. 7 shows first and second buckling modes of the shell with cut-out of 5 mm diameter. The first one is symmetrical and the second one is similar to the assumed deflection function. The symmetrical deflection was rarely observed in the experiment. Values of critical stress corresponding to the first buckling mode are given in Tab. 4.



**Figure 7** Graph of the deflection function near circular cut-out

**Table 4** Critical stress of the shells obtained numerically (FEM)

$R_0$ [mm]	$\sigma_{kr}$ [MPa]		
	steel	brass	aluminium
5	606	406	215

Finite element analysis was performed for ideal structure, so critical stress obtained numerically has greater value than the determined experimentally. An additional detailed nonlinear buckling analysis with taking into account geometrical imperfections is necessary.

The main conclusions:

- Geometrical imperfections have a particularly strong influence on the critical stress value for shells with small cut-out. For the shell with cut-out of a diameter less than 5 mm other imperfections have greater influence on the critical load than the cut-out.
- Numerical investigations should be extended to non-linear model.

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