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On Thermoelasticity in FGL – Tolerance Averaging Technique

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In this paper the problem of linear thermoelasticity in a laminate with functional gradation of properties is considered. In micro level this laminate is made of two different materials, microlaminas, distributed non-periodically but also not randomly along one of directions, what in macro level results in aforementioned functionally gradation of laminate properties. In order to describe behavior of such structure, equations of two models are here presented – the tolerance and the tolerance-asymptotic model. Both are obtained by the tolerance averaging technique. The basic aim of this work is to analyse the influence of some terms from these averaged equations on the distribution and the values of the displacements and the temperature functions. To solve the equations of two proposed models the finite difference method is used.

Keywords: functionally graded laminates, thermoelasticity, tolerance modelling, finite difference method.

1. Introduction

In this work the thermoelasticity issue in functionally graded laminate [1] is considered. The cells of this laminate are composed of two sublayers of different components. The thickness of the cells, called the microstructure parameter is constant and denoted by λ, what is shown in Fig. 1. Therefore the microstructure is realized as uniform distribution of the cells (λ =const.). The macroscopic properties of these structure are changing continuously along direction x_1, normal to the laminas.

In reference to the functionally graded laminates (FGL) the basic cell cannot be defined in a simply way, so various phenomena related to these structures are considered in relation to micromechanical models with idealized geometry. To analyze the laminates, where the distribution function of material properties is non-periodic, the assumptions of idealization analogous to those for periodic composites, can be used.

Among all methods, applied to the macroscopically homogeneous structures, asymptotic homogenization [2-4] and homogenization based on the microlocal parameters [5] should be mentioned. These methods can be modified and adopted to description of structures with functional gradation of properties.

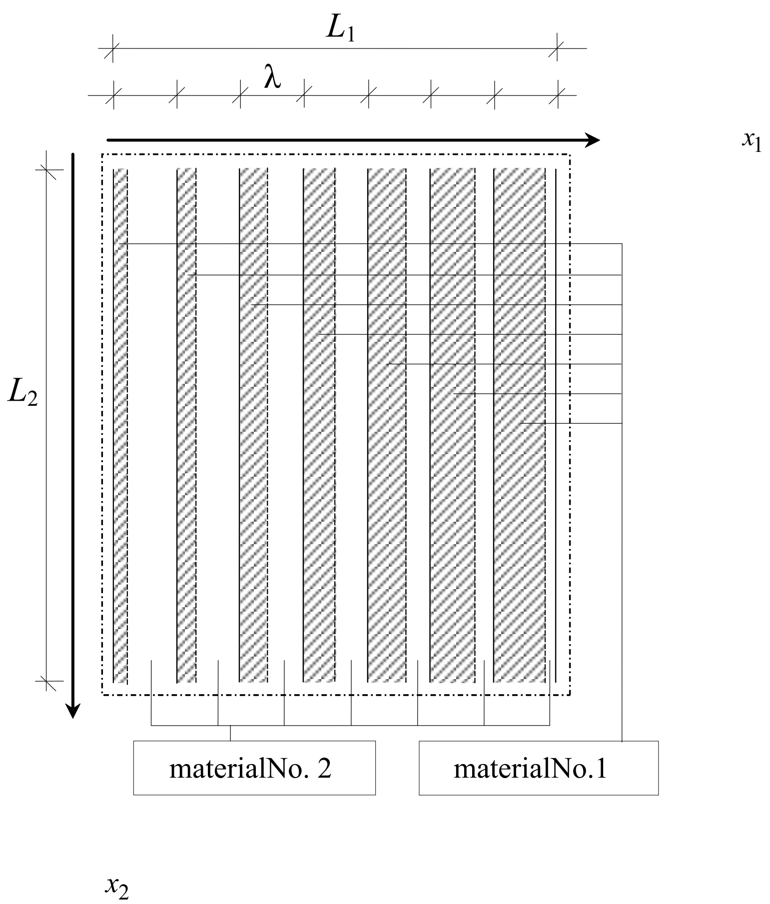

Figure 1 The cross-section of considered laminate

As alternative methods, which can be used in the analysis of functionally graded structures, we can mention the higher order theory [6] and the boundary element method [7].

Unfortunately, most of proposed approaches do not take into account the effect of the microstructure size on the overall behavior of considered laminates. In order to take into account this impact, the tolerance averaging technique is used [8-9].

This technique is expanded and applied in many publications to analyze various issues concerning periodic as well as functionally graded structures. Among them are thermoelasticity problems [10-12], thermal issues [13-15], dynamic [16-20] and stability problems [21].

Moreover in the analysis of various issues related to the composites and layered structures the finite element method [22] or layerwise theory and a differential quadrature finite element method [23] can be used.

The basic aim of this work is to obtain and to present averaged equations for two distinct models, the tolerance and the tolerance-asymptotic one, describing linear thermoelasticity in transversally graded laminates and including the terms responsible for the full connection between the temperature and displacement field. Additional goals are to analyze the influence of these terms on the distribution and values of sought unknowns and to take into account the effect of the microstructure in considered issue.

2. Modelling foundations and procedures

The thermoelasticity issue for the functionally graded laminates can be described by the known Eqs. (1):

$$\begin{aligned} &\nabla \cdot (\mathrm{C} : \nabla \mathbf{u} - \mathbf{B}\theta) - \rho \ddot{\mathbf{u}} = 0 \\ &\nabla \cdot (\mathbf{K} \cdot \nabla \theta) - c\rho \dot{\theta} = T_0 \mathbf{B} : \nabla \dot{\mathbf{u}} \end{aligned} \tag{1}$$

where the vector of total displacements is denoted by $\mathbf{u} = (u_1,\, u_2,\, u_3)^{\mathrm{T}}$, the total temperature is denoted by θ. Tensor of elasticity $\mathbf{C}$ (wherein components are denoted by C_{ijkl}), tensor of conductivity $\mathbf{K}$ (wherein components are denoted by k_{ij}), tensor of thermal extensions $\mathbf{B}$ (wherein components are denoted by b_{ij}), mass density ρ and specific heat c are non-continuous, highly-oscillating and tolerance-periodic material properties, where indices i, j, k, l run over 1, 2, 3.

The main aim of the application of the tolerance averaging technique is to replace the system of differential equations (1) with highly-oscillating, tolerance-periodic and non-continuous coefficients, by equations, where the coefficients are slowly-varying. This technique is based on many concepts, and among them are the averaging operation, tolerance-periodic, slowly-varying and highly-oscillating functions.

By $\partial^i f$ the gradient of the function f is denoted, where i takes values 0, 1, 2 and $\Pi = \Omega \times \Xi$ is a bounded area included in R^3. Coordinates in $\Omega \in \mathrm{R}$ are denoted by $x = x_1$ or $z = z_1$, while in Ξ are denoted by $\varsigma = (\varsigma_1,\, \varsigma_2)$, where Ξ is an area included in R^2. The basic cell is defined as $\Delta {\equiv} (-\lambda/2,\, \lambda/2)$ and $\Delta(x) = x + \Delta$ is a cell with the centre in $x \in \mathrm{R}$.

The averaging operator is defined by Eq. (2):

$$< \partial^i f > (x) \equiv \frac{1}{|\Delta|} \int_{\Delta(x)} \tilde{f}^{(i)}(x,z) dz \tag{2}$$

where $z \in \Delta(x)$ and by $\tilde{f}^{(i)}(x, \cdot\,)$ a periodic approximation of the gradient $\partial^i f()$ in $\Delta(x)$ is denoted.

If the terms given by (3) are fulfilled, then function $f \in H^r(\Omega)$ can be called the tolerance periodic-function, in reference to the basic cell Δ and tolerance parameter δ:

$$\begin{array}{l}(\forall x \in \Omega)\left(\exists \tilde{f}^{(i)}(x, \cdot\,) \in H^0(\Delta)\right) \quad \left(\left\|\partial^i f(\,\cdot\,) - \;\tilde{f}^{(i)}(x, \cdot\,)\right\|_{H^0(\Omega_x)} \leq \delta\right) \\ \int_{\Delta(\cdot)} \tilde{f}^{(i)}(\,\cdot\,,z)dz \in C^0(\bar{\Omega})\end{array} \tag{3}$$

for every $i = 0, 1, \ldots, r$, Ω_x is defined by Eq. (4):

$$\Omega_x = \Omega \cap \bigcup_{z \in \Delta(x)} \Delta\left(z\right), \, x \in \overline{\Omega} \tag{4}$$

and $H^0(\Delta)$ is a space of Δ-periodic functions, which are square integrable.

Continuous function u can be called the slowly-varying function, in reference to the basic cell Δ and tolerance parameter δ, if it is a tolerance-periodic function and the condition given by Eq. (5) is fulfilled:

$$(\forall x \in \Omega)\left(\tilde{u}^{(i)}\left(x, \cdot\right)\big|_{\Delta(x)} = \partial^i u(x)\right) \tag{5}$$

In other words, periodic approximation of $\partial^i u(\cdot)$ in $\Delta(x)$ for every $x \epsilon \Omega$ is a constant function.

Essentially bounded function h can be called the highly-oscillating function, in reference to the basic cell Δ and tolerance parameter δ, if Eq. (6) is satisfied:

$$(\forall x \in \Omega)\left(\tilde{h}^{(i)}\left(x, \cdot\right)\big|_{\Delta(x)} = \partial^i \tilde{h}(x)\right) \tag{6}$$

and h is a tolerance-periodic function.

The tolerance modelling is based on two main assumptions. The first one is the micro-macro decomposition, where the basic unknowns (the temperature and the displacements field) can be taken as sums of the averaged and the oscillating parts, according to Eqs. (7):

$$\begin{array}{l} u_i\left(x,\varsigma,t\right) = w_i\left(x,\varsigma,t\right) + h^m\left(x\right) v_{im}\left(x,\varsigma,t\right) \\ \theta\left(x,\varsigma,t\right) = \vartheta\left(x,\varsigma,t\right) + g^a\left(x\right) \psi_a\left(x,\varsigma,t\right) \end{array} \tag{7}$$

The oscillating part can be expressed as sums of products of known fluctuation shape functions and the fluctuation amplitudes, which are the new basic unknowns. The macrodisplacements w_i, i takes values 1, 2, 3, and the macrotemperature ϑ – the basic unknowns and the fluctuation amplitudes of the displacements V_{im} and the temperature ψ_a – the new basic unknowns are the slowly-varying functions of the coordinate x_1. The fluctuation shape functions of the displacements and the temperature, denoted by h^m and g^a, respectively, have to be defined for each analyzed case where $a = 1, \ldots, N$ and $m = 1, \ldots, N$. In this note one fluctuation shape function is only assumed ($N = 1$), namely the saw type function. This fluctuation shape function guarantees the continuity of the displacements and the temperature between the layers and between the sublayers.

The second assumption in tolerance modelling is the periodic approximation of kth derivatives of functions of displacements and temperature, which can be defined for $k \geq 1$ according to Eqs. (8):

$$\begin{aligned}\tilde{u}_i^{(k)}(x,z,\varsigma) &= \nabla^k w_i(x,\varsigma) + \partial^k \tilde{h}^m(x,z)\, v_{im}(x,\varsigma) + \tilde{h}^m(x,z)\,\bar{\nabla}^k v_{im}(x,\varsigma)\\ \tilde{\theta}^{(k)}(x,z,\varsigma) &= \nabla^k \vartheta(x,\varsigma) + \partial^k \tilde{g}^a(x,z)\,\psi_a(x,\varsigma) + \tilde{g}^a(x,z)\,\bar{\nabla}^k \psi_a(x,\varsigma)\end{aligned} \tag{8}$$

where $z \in \Delta(x)$, $x \in \Omega$ and $\varsigma \in \Xi$.

Averaged equations of tolerance model are obtained by using the extended stationary action principle, where the action functional is defined according to Eq. (9):

$$J(\theta,\mathbf{u}) = \int_{t_0}^{t_1} \int_{\Omega} L(x,t,\theta,\nabla\theta,\mathbf{u},\dot{\mathbf{u}},\nabla\mathbf{u},\mathbf{p},p)\; d\Omega dt \tag{9}$$

wherein Lagrangian L is assumed in the form of Eq. (10):

$$L = \frac{1}{2}(\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} - \nabla\mathbf{u} : \mathrm{C} : \nabla\mathbf{u} + \nabla\theta\cdot\mathbf{K}\cdot\nabla\theta) + \mathbf{p}\cdot\mathbf{u} + p\theta \tag{10}$$

Parameters $\mathbf{p}$ and p are in the form of Eqs. (11):

$$\begin{aligned}\mathbf{p} &= -\nabla\cdot(\mathbf{B}\theta)\\ p &= c\rho\dot{\theta} + T_0\mathbf{B} : \nabla\dot{\mathbf{u}}\end{aligned} \tag{11}$$

and represent invariational constitutive laws.

Then by using the assumptions (7) and (8), the action functional can be redefined, following Eq. (12):

$$\begin{aligned}&J(\vartheta,\psi_a,\mathbf{w},\mathbf{v}_m)\\ &= \int_{t_0}^{t_1}\int_{\Omega} L_{gh}\begin{pmatrix} x,t,\vartheta,\nabla\vartheta,\psi_a,\overline{\nabla}\psi_a,\mathbf{w},\dot{\mathbf{w}},\nabla\mathbf{w},\mathbf{v}_m,\dot{\mathbf{v}}_m,\overline{\nabla}\mathbf{v}_m,\\ \langle p\rangle,\langle pg^a\rangle,\langle\mathbf{p}\rangle,\langle\mathbf{p}h^m\rangle\end{pmatrix} d\Omega dt\end{aligned} \tag{12}$$

and by doing appropriate averaging and transformations, Lagrangian L_{gh} can be defined in the form of Eq. (13):

$$\begin{aligned}L_{gh} &= \frac{1}{2}(\langle\rho\rangle\,\dot{\mathbf{w}}\cdot\dot{\mathbf{w}} + \langle\rho h^m h^n\rangle\,\dot{\mathbf{v}}_m\cdot\dot{\mathbf{v}}_n - \nabla\mathbf{w} : \langle\mathrm{C}\rangle : \nabla\mathbf{w}\\ &-\mathbf{v}_m\cdot\langle\partial h^m\cdot\mathrm{C}\cdot\partial h^n\rangle\cdot\mathbf{v}_n)\\ &-\frac{1}{2}\left(\overline{\nabla}\mathbf{v}_m : \langle\mathrm{C}h^m h^n\rangle : \overline{\nabla}\mathbf{v}_m - \nabla\vartheta\cdot\langle\mathbf{K}\rangle\cdot\nabla\vartheta - \psi_a\left\langle\partial g^a\cdot\mathbf{K}\cdot\partial g^b\right\rangle\psi_b\right)\\ &+\frac{1}{2}\left(\overline{\nabla}\psi_a\cdot\left\langle\mathbf{K}g^a g^b\right\rangle\cdot\overline{\nabla}\psi_b\right) - \nabla\mathbf{w} : \langle\mathrm{C}\cdot\partial h^m\rangle\cdot\mathbf{v}_m - \nabla\mathbf{w} : \langle\mathrm{C}h^m\rangle : \overline{\nabla}\mathbf{v}_m\\ &-\overline{\nabla}\mathbf{v}_m : \langle\mathrm{C}h^m\cdot\partial h^n\rangle\cdot\mathbf{v}_n + \nabla\vartheta\cdot\langle\mathbf{K}\partial g^a\rangle\,\psi_a + \nabla\vartheta\cdot\langle\mathbf{K}g^a\rangle\cdot\overline{\nabla}\psi_a\\ &+\overline{\nabla}\psi_a\cdot\left\langle\mathbf{K}g^a\cdot\partial g^b\right\rangle\psi_b + \langle\mathbf{p}_g\rangle\cdot\mathbf{w} + \langle\mathbf{p}_g h^m\rangle\cdot\mathbf{v}_m + \langle\mathbf{p}_{gh}\rangle\cdot\vartheta + \langle\mathbf{p}_{gh}g^a\rangle\cdot\psi_a\end{aligned} \tag{13}$$

where $\mathbf{p}_g$ and p_{gh} – modified invariational constitutive laws, are defined in the form of Eqs. (14) and (15)

$$\langle\mathbf{p}_g\rangle = -\langle\nabla\cdot\mathbf{B}\rangle\,\vartheta - \langle\mathbf{B}\rangle\cdot\nabla\vartheta - \langle(\nabla\cdot\mathbf{B})\,g^a\rangle\,\psi_a - \langle\mathbf{B}\cdot\partial g^a\rangle\,\psi_a - \langle\mathbf{B}g^a\rangle\cdot\overline{\nabla}\psi_a \tag{14}$$

$$\langle p_{gh}\rangle = \langle c\rho\rangle\,\dot{\vartheta} + \langle T_0\mathbf{B}\rangle : \nabla\dot{w} + \langle T_0\mathbf{B}\cdot\partial h^m\rangle : \dot{v}_m + \langle T_0\mathbf{B}h^m\rangle : \overline{\nabla}\dot{v}_m \tag{15}$$

and by assuming that the variation of functional $\delta J(\vartheta,\psi_a,\ \boldsymbol{w},\boldsymbol{v}_m)$ is equal to zero, the variation δL_{gh} is calculated, which leads to Eqs. (16):

$$\begin{aligned}
&\frac{\partial L_{gh}}{\partial\vartheta} - \nabla\cdot\left(\frac{\partial L_{gh}}{\partial\nabla\vartheta}\right) = 0\\
&\frac{\partial L_{gh}}{\partial\psi_a} - \overline{\nabla}\cdot\left(\frac{\partial L_{gh}}{\partial\overline{\nabla}\psi_a}\right) = 0\\
&\frac{\partial L_{gh}}{\partial\mathbf{w}} - \frac{\partial}{\partial t}\left(\frac{\partial L_{gh}}{\partial\dot{\mathbf{w}}}\right) - \nabla\cdot\left(\frac{\partial L_{gh}}{\partial\nabla\mathbf{w}}\right) = 0\\
&\frac{\partial L_{gh}}{\partial\mathbf{v}_m} - \frac{\partial}{\partial t}\left(\frac{\partial L_{gh}}{\partial\dot{\mathbf{v}}_m}\right) - \overline{\nabla}\cdot\left(\frac{\partial L_{gh}}{\partial\overline{\nabla}\mathbf{v}_m}\right) = 0
\end{aligned}\tag{16}$$

called the Euler-Lagrange equations. Then the derivatives of Lagrangian L_{gh} are calculated and combining with these equations, Eqs. (17) are obtained:

$$\begin{aligned}
&\langle c\rho\rangle\,\dot{\vartheta} - \nabla\cdot\big(\langle\mathbf{K}\rangle\cdot\nabla\vartheta + \langle\mathbf{K}\cdot\partial g^a\rangle\,\psi_a + \langle\mathbf{K}g^a\rangle\cdot\overline{\nabla}\psi_a\big)\\
&= -\underline{\underline{\langle T_0\mathbf{B}\rangle : \nabla\dot{\mathbf{w}}}} - \underline{\underline{\langle T_0\mathbf{B}\cdot\partial h^m\rangle\cdot\dot{\mathbf{v}}_m}} - \underline{\underline{\langle T_0\mathbf{B}\cdot h^m\rangle : \overline{\nabla}\mathbf{v}_m}}\\
&\left\langle c\rho g^a g^b\right\rangle\dot{\psi}_b - \overline{\nabla}\cdot\big(\left\langle\mathbf{K}g^a g^b\right\rangle\overline{\nabla}\psi_b + \langle\mathbf{K}g^a\rangle\cdot\nabla\vartheta + \left\langle\mathbf{K}g^a\partial g^b\right\rangle\psi_b\big)\\
&+\left\langle\partial g^a\cdot\mathbf{K}\cdot\partial g^b\right\rangle\psi_b + \langle\mathbf{K}\cdot\partial g^a\rangle\cdot\nabla\vartheta + \left\langle\mathbf{K}g^b\partial g^a\right\rangle\cdot\overline{\nabla}\psi_b\\
&= -\underline{\underline{\langle T_0\mathbf{B}g^a\rangle : \nabla\dot{\mathbf{w}}}} - \underline{\underline{\langle T_0\mathbf{B}g^a\cdot\partial h^m\rangle\cdot\dot{\mathbf{v}}_m}} - \underline{\underline{\langle T_0\mathbf{B}g^a h^m\rangle : \overline{\nabla}\dot{\mathbf{v}}_m}}\\
&\nabla\cdot\big(\langle \mathrm{C}\rangle : \nabla\mathbf{w} + \langle\mathrm{C}\cdot\partial h^m\rangle\cdot\mathbf{v}_m + \langle\mathrm{C}h^m\rangle : \overline{\nabla}\mathbf{v}_m\big) = \langle\rho\rangle\,\ddot{\mathbf{w}} + \langle\mathbf{B}\rangle\cdot\nabla\vartheta\\
&+\langle\mathbf{B}\cdot\partial g^a\rangle\,\psi_a + \langle\mathbf{B}g^a\rangle\cdot\overline{\nabla}\psi_a + \langle\nabla\cdot\mathbf{B}\rangle\,\vartheta + \langle(\nabla\cdot\mathbf{B})\,g^a\rangle\,\psi_a\\
&\overline{\nabla}\cdot\big(\langle\mathrm{C}h^m h^n\rangle : \overline{\nabla}\mathbf{v}_n + \langle\mathrm{C}h^m\rangle : \nabla\mathbf{w} + \langle\mathrm{C}h^m\cdot\partial h^n\rangle\cdot\mathbf{v}_n\big) - \langle\partial h^m\cdot\mathrm{C}\cdot\partial h^n\rangle\cdot\mathbf{v}_n\\
&-\langle\mathrm{C}\cdot\partial h^m\rangle : \nabla\mathbf{w} - \langle\mathrm{C}h^n\cdot\partial h^m\rangle : \overline{\nabla}\mathbf{v}_n = \langle\rho h^m h^n\rangle\,\ddot{\mathbf{v}}_n + \langle\mathbf{B}h^m\rangle\cdot\nabla\vartheta\\
&+\langle\mathbf{B}h^m\cdot\partial g^a\rangle\,\psi_a + \langle\mathbf{B}g^a h^m\rangle\cdot\overline{\nabla}\psi_a + \langle(\nabla\cdot\mathbf{B})\,h^m\rangle\,\vartheta + \langle(\nabla\cdot\mathbf{B})\,g^a h^m\rangle\,\psi_a
\end{aligned}\tag{17}$$

which are the tolerance model equations, where the double-underlined terms are responsible for the full connection between the temperature and the displacements.

Directly from Eqs. (17), by omitting the terms of order of microstructure parameter λ, the equations of the tolerance-asymptotic model are obtained in the form of Eqs. (18):

$$\begin{aligned}
&\langle c\rho\rangle\,\dot{\vartheta} - \nabla\cdot(\langle\mathbf{K}\rangle\cdot\nabla\vartheta + \langle\mathbf{K}\cdot\partial g^a\rangle\,\psi_a) = -\underline{\underline{\langle T_0\mathbf{B}\rangle : \nabla\dot{\mathbf{w}}}} - \underline{\underline{\langle T_0\mathbf{B}\cdot\partial h^m\rangle\cdot\dot{\mathbf{v}}_m}}\\
&\left\langle\partial g^a\cdot\mathbf{K}\cdot\partial g^b\right\rangle\psi_b + \langle\mathbf{K}\cdot\partial g^a\rangle\cdot\nabla\vartheta = 0\\
&\nabla\cdot(\langle\mathrm{C}\rangle : \nabla\mathbf{w} + \langle\mathrm{C}\cdot\partial h^m\rangle\cdot\mathbf{v}_m) = \langle\rho\rangle\,\ddot{\mathbf{w}} + \langle\mathbf{B}\rangle\cdot\nabla\vartheta + \langle\mathbf{B}\cdot\partial g^a\rangle\,\psi_a + \langle\nabla\cdot\mathbf{B}\rangle\,\vartheta\\
&\langle\partial h^m\cdot\mathrm{C}\cdot\partial h^n\rangle\cdot\mathbf{v}_n + \langle\mathrm{C}\cdot\partial h^m\rangle : \nabla\mathbf{w} = 0
\end{aligned}\tag{18}$$

which omit the influence of the microstructure size. The double underlined terms in (18) are responsible for fully coupling of temperature and displacement field.

3. Example

The problem under consideration was non-stationary issue of two-dimensional thermoelasticity in laminate, cf. Fig. 1, made of two different materials.

For both sublayers the material properties such as Young's modules E, Poisson's ratios υ, mass densities ρ, expansion coefficients α, specific heats c and components of tensor of conductivity k_{ij}were defined. Based on these properties the Lame's constants Λ and μ, as components of tensor of elasticity C_{ijkl} and components of tensor of thermal extensions b_{ij}, were calculated, where i, j, k, l = 1, 2, 3. The trigonometric distribution function of material properties was assumed according to Eq. (19):

$$\mathrm{v}_1\left(x_1\right) = \sin\left(\frac{x_1}{L_1}\right) \tag{19}$$

where L_1 was a dimension of the composite along x_1 direction normal to the laminas.

Based on Eqs. (17) and the assumption of the plane strain state, the equations of the tolerance model for this specific issue were obtained in the form of Eqs. (20):

$$\begin{aligned}
&\partial_1\left(< C_{1111} > \partial_1 w_1 + < C_{1122} > \partial_2 w_2 + < C_{1111}\partial h > v_1\right)\\
&-\left(< b_{11} > \partial_1\vartheta + < b_{11}\partial g > \psi\right)\\
&= -\partial_2\left(< C_{1221} > \partial_1 w_2 + < C_{1212} > \partial_2 w_1 + < C_{1221}\partial h > v_2\right) + < \rho > \ddot{w}_1\\
&\partial_1\left(< C_{2121} > \partial_1 w_2 + < C_{2112} > \partial_2 w_1 + < C_{2121}\partial h > v_2\right)\\
&+\partial_2\left(< C_{2211} > \partial_1 w_1\right)\\
&= -\partial_2\left(< C_{2222} > \partial_2 w_2 + < C_{2211}\partial h > v_1\right) + < b_{22} > \partial_2\vartheta + < \rho > \ddot{w}_2\\
&\partial_2\left(< C_{1212}hh > \partial_2 v\right) - < C_{1111}\partial h > \partial_1 w_1 - < C_{1122}\partial h > \partial_2 w_2\\
&= < C_{1111}\partial h\partial h > v_1 + < \rho hh > \ddot{v}_1\\
&\partial_2\left(< C_{2222}hh > \partial_2 v_2\right) - < b_{22}gh > \partial_2\psi - < C_{2121}\partial h > \partial_1 w_2\\
&= < C_{2112}\partial h > \partial_2 w_1 + < C_{2121}\partial h\partial h > v_2 + < \rho hh > \ddot{v}_2\\
&\partial_1\left(< k_{11} > \partial_1\vartheta + < k_{11}\partial g > \psi\right) + \partial_2\left(< k_{22} > \partial_2\vartheta\right) - < c\rho > \dot{\vartheta}\\
&= \underline{\underline{< T_0 b_{11} > \partial_1\dot{w}_1}} + \underline{\underline{< T_0 b_{22} > \partial_2\dot{w}_2}} + \underline{\underline{< T_0 b_{11}\partial h > \dot{v}_1}}\\
&\partial_2\left(< k_{22}gg > \partial_2\psi\right) - < k_{11}\partial g > \partial_1\vartheta - < k_{11}\partial g\partial g > \psi\\
&= < c\rho gg > \dot{\psi} + \underline{\underline{< T_0 b_{22}gh > \partial_2\dot{v}_2}}
\end{aligned} \tag{20}$$

and directly from these equations, by neglecting the terms dependent explicitly on the microstructure parameter λ (of order λ), the tolerance-asymptotic model equations were obtained, according to Eqs. (21):

$$\begin{aligned}
&\partial_1\left(< C_{1111} > \partial_1 w_1 + < C_{1122} > \partial_2 w_2 + < C_{1111}\partial h > v_1\right)\\
&-\left(< b_{11} > \partial_1\vartheta + < b_{11}\partial g > \psi\right)\\
&= -\partial_2\left(< C_{1221} > \partial_1 w_2 + < C_{1212} > \partial_2 w_1 + < C_{1221}\partial h > v_2\right) + < \rho > \ddot{w}_1\\
&\partial_1\left(< C_{2121} > \partial_1 w_2 + < C_{2112} > \partial_2 w_1 + < C_{2121}\partial h > v_2\right) + \partial_2\left(< C_{2211} > \partial_1 w_1\right)\\
&= -\partial_2\left(< C_{2222} > \partial_2 w_2 + < C_{2211}\partial h > v_1\right) + < b_{22} > \partial_2\vartheta + < \rho > \ddot{w}_2\\
&- < C_{1111}\partial h > \partial_1 w_1 - < C_{1122}\partial h > \partial_2 w_2 - < C_{1111}\partial h\partial h > v_1 = 0\\
&- < C_{2121}\partial h > \partial_1 w_2 - < C_{2112}\partial h > \partial_2 w_1 + < C_{2121}\partial h\partial h > v_2 = 0\\
&\partial_1\left(< k_{11} > \partial_1\vartheta + < k_{11}\partial g > \psi\right) + \partial_2\left(< k_{22} > \partial_2\vartheta\right) - < c\rho > \dot{\vartheta}\\
&= \underline{\underline{< T_0 b_{11} > \partial_1\dot{w}_1}} + \underline{\underline{< T_0 b_{22} > \partial_2\dot{w}_2}} + \underline{\underline{< T_0 b_{11}\partial h > \dot{v}_1}}\\
&- < k_{11}\partial g > \partial_1\vartheta - < k_{11}\partial g\partial g > \psi = 0
\end{aligned} \tag{21}$$

where the double-underlined terms were responsible for the full connection between sought unknowns.

The known temperature as external load was assumed on the edges of the laminate according to Eqs (22):

$$\theta|_{x_1=0} = \theta_l, \quad \theta|_{x_1=L_1} = \theta_r, \quad \theta|_{x_2=0} = \theta_t, \quad \theta|_{x_2=L_2} = \theta_b \tag{22}$$

The known displacements along x_1 direction following to Eqs. (23):

$$u_1|_{x_1=0} = u_{1l}, \quad u_1|_{x_1=L_1} = u_{1r}, \quad u_1|_{x_2=0} = u_{1t}, \quad u_1|_{x_2=L_2} = u_{1b} \tag{23}$$

and the known displacements along x_2 direction following to Eqs. (24):

$$u_2|_{x_1=0} = u_{2l}, \quad u_2|_{x_1=L_1} = u_{2r}, \quad u_2|_{x_2=0} = u_{2t}, \quad u_2|_{x_2=L_2} = u_{2b} \tag{24}$$

as boundary conditions were assumed. Then the appropriate boundary conditions for other unknowns and necessary initial conditions were defined.

To solve Eqs. (20) and (21) the finite difference method was used. Along direction parallel to the laminas the grid nodes distribution was uniform and along direction perpendicular to the laminas the nodes were defined in the middle of every sublayer, between the sublayers and between the cells.

By using the finite difference method in each of presented models the set of non-homogeneous discretized equations was obtained with the macrodisplacements, the macrotemperature and the fluctuation amplitudes of the displacements and the temperature as a unknowns, according to Eq. (25):

$$(1-\alpha)\,\mathbf{K}\mathbf{q}^{k+1} + \alpha \mathrm{K}\mathrm{q}^{k} + \mathbf{K}_t\mathbf{q}^{k+1} + (1-\alpha)\,\mathbf{Q}^{k+1} + \alpha \mathrm{Q}^{k} = 0 \tag{25}$$

where $\mathbf{K}$ and $\mathbf{K}_t$ was matrix of coefficients respectively independent and dependent of time coordinate in individual equations, $\mathbf{Q}$ was a vector of free terms, $\mathbf{q}$ – vector of unknowns ordered alternately at individual points, k defines specific point in time and α is a parameter determining the approach in the context of numerical methods. In this note α is equal to 0.5, because the Crank-Nicolson method was used.

The obtained results were shown in Figs. 2-10 in the form of plots of dimensionless values of sought unknowns. The dimensionless values of the macrotemperature T, the fluctuation amplitudes of the temperature Ψ, the total temperature Θ, the macrodisplacements W_1, the fluctuation amplitudes of the displacements V_1, the total displacements U_1, the macrodisplacements W_2, the fluctuation amplitudes of the displacements V_2 and the total displacements U_2, were obtained by dividing the received values by the maximum worth of individual unknown in the asymptotic model. The plots were made for the specific dimensionless time coordinate $\tau_f = t/1$ [s] and the dimensionless spatial coordinates $\xi_1 = x_1 L_1$ and $\xi_2 = x_2/L_2$. Then the results obtained by solving the equations of both models were compared with the results obtained by solving the same equations but after omitting the double-underlined terms, responsible for the full connection between the temperature and the displacements.

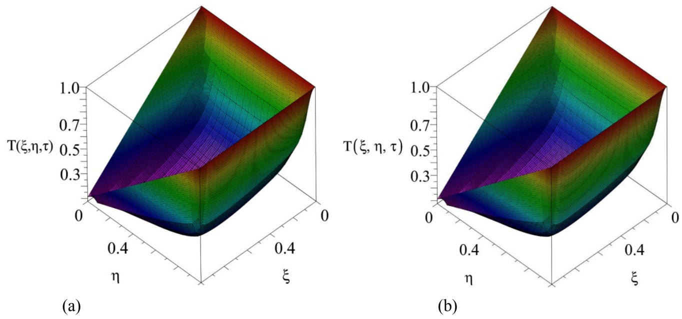

Figure 2 The dimensionless values of the macrotemperature (a) in the asymptotic and (b) in the tolerance model

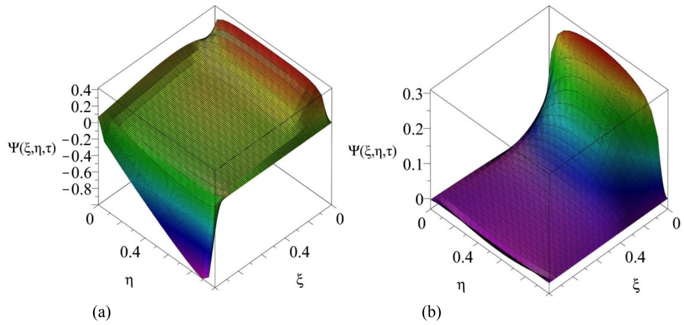

Figure 3 The dimensionless values of the fluctuation amplitudes of the temperature (a) in the asymptotic and (b) in the tolerance model

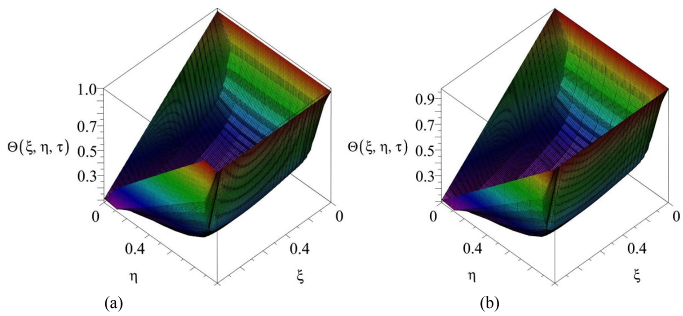

Figure 4 The dimensionless values of the total temperature (a) in the asymptotic and (b) in the tolerance model

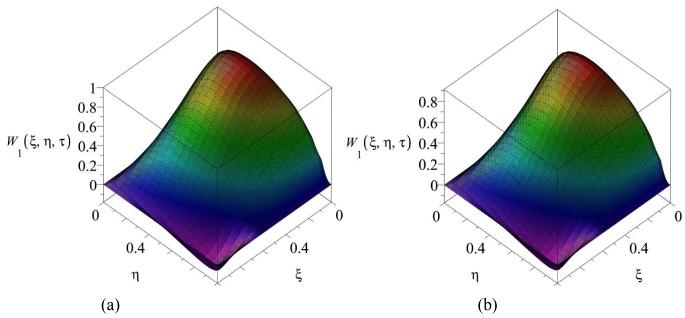

Figure 5 The dimensionless values of the macrodisplacements along direction x_1 (a) in the asymptotic and (b) in the tolerance model

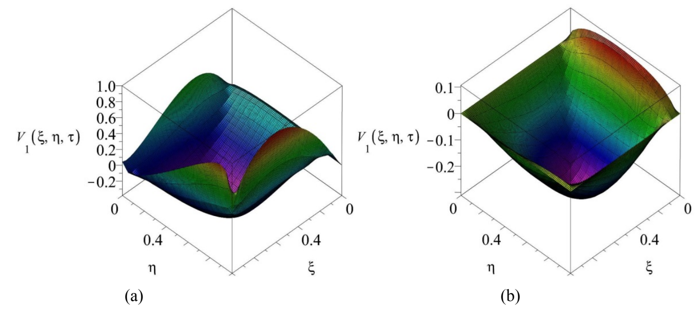

Figure 6 The dimensionless values of the fluctuation amplitudes of the displacements along direction x_1 (a) in the asymptotic and (b) in the tolerance model

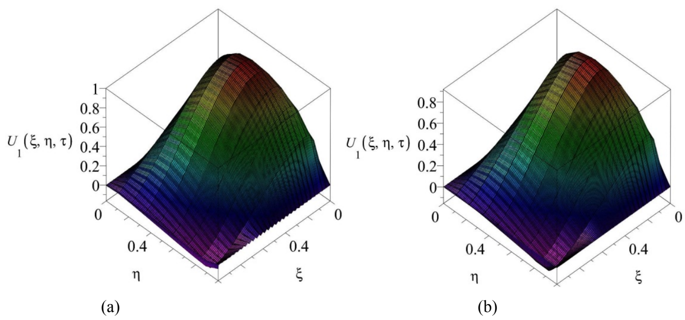

Figure 7 The dimensionless values of the total displacements along direction x_1 (a) in the asymptotic and (b) in the tolerance model

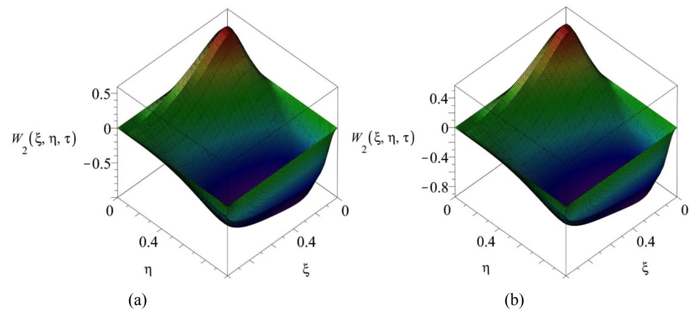

Figure 8 The dimensionless values of the macrodisplacements along direction x_2 (a) in the asymptotic and (b) in the tolerance model

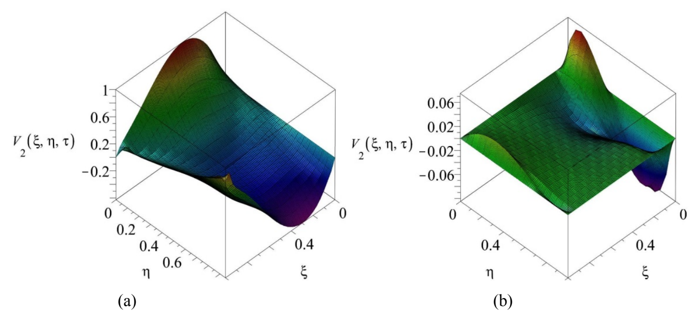

Figure 9 The dimensionless values of the fluctuation amplitudes of the displacements along direction x_2 (a) in the asymptotic and (b) in the tolerance model

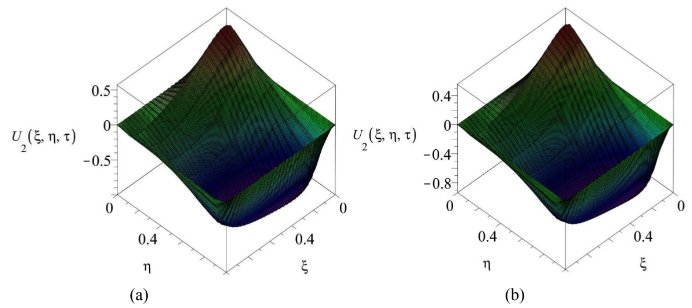

Figure 10 The dimensionless values of the total displacements along direction x_2 (a) in the asymptotic and (b) in the tolerance model

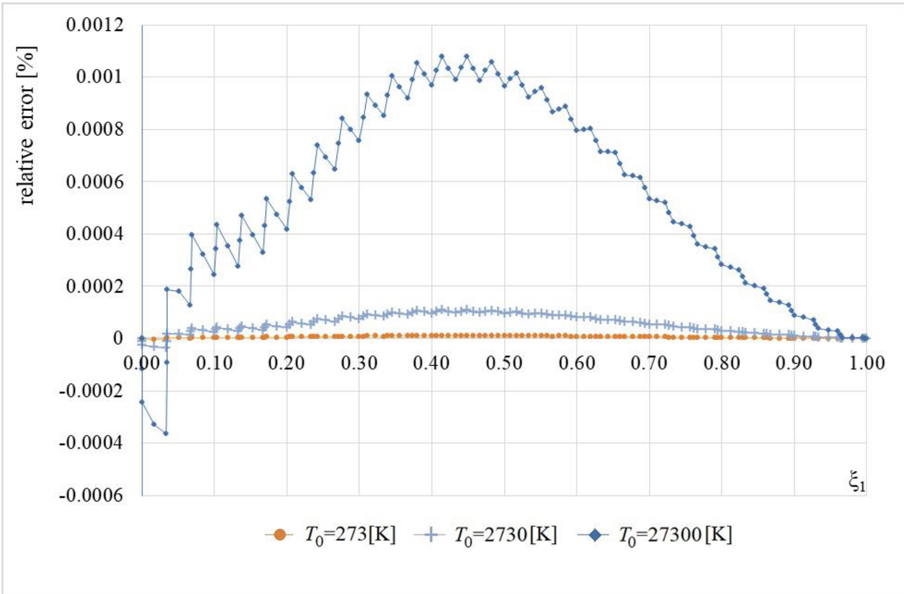

Figure 11 The relative error of the dimensionless values of the total temperature

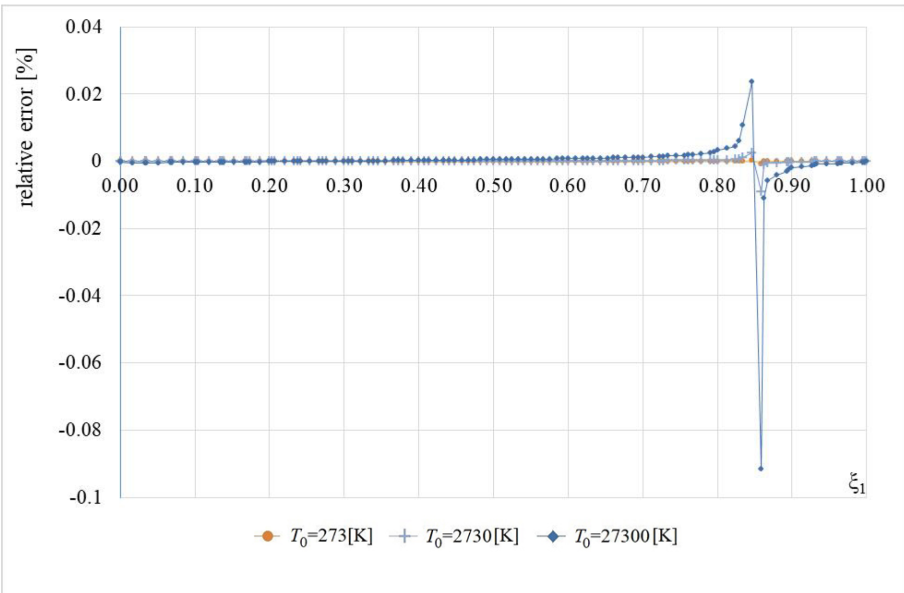

Figure 12 The relative error of the dimensionless values of the total displacements along direction x_1

The comparison of the results (the relative error between the values of unknowns while solving the equations including the double-underlined terms and the equations without these terms) was shown in Figs. 11-13. By line with markers in the form of dots, pluses and rhombs it was denoted the relative error defined, on the example of the total temperature, by Eq. (26), Eq. (27) and Eq. (28), respectively.

$$\frac{\Theta|_{T_0=0[\mathrm{K}]} - \Theta|_{T_0=273[\mathrm{K}]}}{\Theta|_{T_0=0[\mathrm{K}]}} 100\% \tag{26}$$

$$\frac{\Theta|_{T_0=0[\mathrm{K}]} - \Theta|_{T_0=2730[\mathrm{K}]}}{\Theta|_{T_0=0[\mathrm{K}]}} 100\% \tag{27}$$

$$\frac{\Theta|_{T_0=0[\mathrm{K}]} - \Theta|_{T_0=27300[\mathrm{K}]}}{\Theta|_{T_0=0[\mathrm{K}]}} 100\% \tag{28}$$

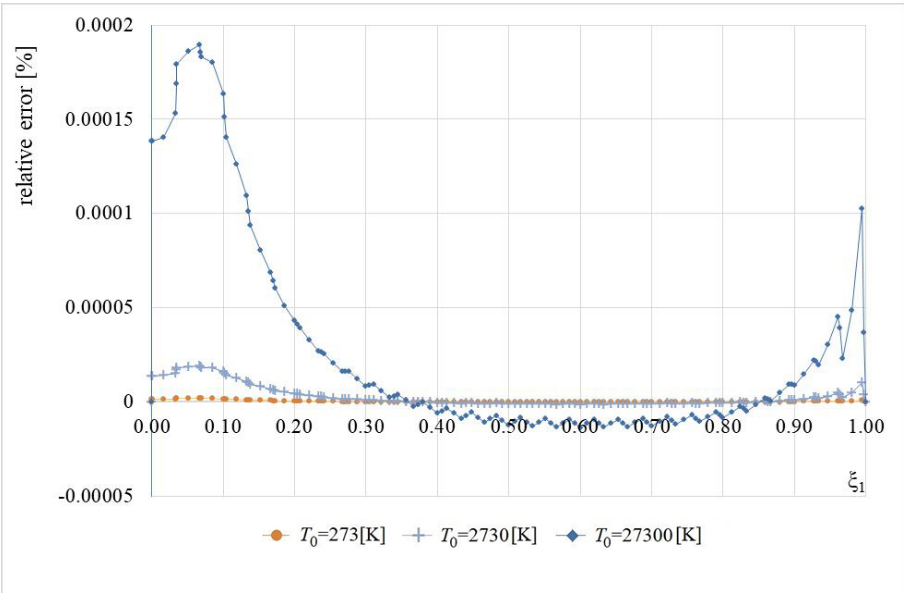

Figure 13 The relative error of the dimensionless values of the total displacements along direction x_2

4. Remarks

Based on the presented considerations and results some remarks can be formulated:

1. by using the tolerance averaging technique it is possible to replace the differential equations with tolerance-periodic, highly-oscillating and non-continuous coefficients, by the set of equations where the coefficients are constant or slowly-varying

2. by using the tolerance modelling it is possible to analyze the whole structure, without necessity of analysis the problem in a single cell

3. by using the equations of the tolerance model it is possible to take into account the effect of the microstructure size in thermolasticity problems, while in the asymptotic model equations this effect is omitted

4. the influence of the microstructure in this issue can be observed as wedges in the figures of the total displacements and the total temperature

5. the equations of two presented models can be used in the analysis of some specific cases, where the distribution of properties is functional, but non-periodic

6. the maximum values of the macrodisplacements, the macrotemperature, the total displacements and the total temperature in analysed models are very similar (in the tolerance-asymptotic model are slightly higher than in the tolerance model)

7. the relative error of the values of unknowns while solving the equations including the terms responsible for the full connection between the displacements and the temperature and the equations without these terms takes values $0 \div 0.1\%$, so the impact of these terms is negligibly small.

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