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Research Article

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Road to chaos in a Duffing oscillator with time delay loop

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Abstract: This article examines a single Duffing oscillator with a time delay loop. The research aims to check the impact of the time delay value on the nature of the solution, in particular the scenario of transition to a chaotic solution. Dynamic tools such as bifurcation diagrams, phase portraits, Poincaré maps, and FFT analysis will be used to evaluate the obtained results.

Keywords: Duffing oscillator, time delay loop, bifurcation diagram, phase portraits, Poincaré maps

1 Introduction

The analysis of nonlinear dynamical systems has been the subject of intense research in recent decades. When examining such systems, rapid changes in the stability of the solution (bifurcations) or irregular solutions, sensitive to initial conditions (chaos), can be observed. The cause of observation of chaotic behaviors in dynamical systems is their property, consisting in the exponential divergence of initially closely related trajectories in the phase space area [1]. The development of research on the theory of chaos has led to the emergence of many new topics such as chaos control [2, 3], synchronization of chaos [4, 5], and roads to chaos.

2 Roads to chaos

The first scenario of the transition of the system from periodic to chaotic behavior was presented by L.D. Landau in 1944 [6].

Four years later (1948), an independent theory was presented by E.A. Hopf [7]. The Landau-Hopf scenario assumes that during the passage of a certain parameter through a critical value (*e.g.*, the Reynolds number (R)), which is a parameter characterizing the flow of fluids, the stationary flow loses its stability. As the Reynolds number increases, new frequencies appear. For $R \rightarrow \infty$, the speed of the formation of new frequencies increases, which leads to the appearance of a wide frequency band, characteristic of chaotic behaviors.

Another similar scenario of the transition to chaos is the Newhouse-Ruelle-Takens scenario [8]. It refers to and corrects the Landau-Hopf scenario.

In 1971, D. Ruelle and F. Takens proved [9] that an infinite series of Hopf bifurcations is not necessary to achieve destabilization of the system. They presented a system in which, after three Hopf bifurcations, the system reaches an orbit that may lose its stability and pass to a strange chaotic attractor.

Another scenario of the transition to chaos was proposed by M.J. Feigenbaum in 1978 [10, 11]. According to him, the road to chaos can be done through period-doubling bifurcation.

The next scenario of the transition to chaos is the Pomeau and Manneville scenario [12, 13] presented in 1980. The research they obtained shows the possibility of transition to chaotic behavior through sudden bifurcations. This transition is related to the occurrence of system intermittency, *i.e.*, a system shift between two types of behavior – almost periodic and chaotic behavior.

Current research very often refers to the above-mentioned scenarios when analyzing nonlinear dynamical systems.

3 Nonlinear dynamical systems with time delay

The research carried out so far shows that the dynamics of systems with the introduced time delay may be very complicated and may have a number of interesting features. In addition, it has been shown that the use of time delay in dynamical systems is one of the effective methods of con-

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trolling (or anti-controlling) chaos because the delay time can be easily controlled and implemented in real applications.

Already in the '70 s, studies in which the introduction of a time delay to the analyzed systems led to very complicated, chaotic behaviors (Mackey and Glass [14], Farmer [15], Lu and He [16], Awrejcewicz and Wojewoda [17]) were presented.

In the following years, A. Maccari [18] presented the effect of the time delay and the feedback gain on the peak amplitude of the fundamental resonance in the nonlinear Van der Pol oscillator. He showed that the selection of appropriate values of the time delay and the feedback gain reduces the value of the peak amplitude and suppresses the quasi-periodic motion.

In the work of P.Yu, Y.Yuan, J.Xu [19] from 2002, a nonlinear oscillator with an introduced time delay to the linear and nonlinear parts of the equation in the feedback loop was presented. By changing the value of the time delay, the rich dynamics of the system were observed.

In 2003, J.Xu and K.W.Chung [20] presented a Van der Pol-Duffing oscillator with a time delay loop introduced to the linear and nonlinear parts of the equation. They were given two roads to the chaotic solution – by period-doubling bifurcation and torus decay bifurcation. Moreover, they recognized that the time delay plays a very important role in the analysis of the behavior of dynamical systems. Appropriate selection of the time delay value effectively damps vibrations. They found that the time delay can be used as a simple "switch" to control the behavior of the system. Thanks to it, it is possible not only to change an unstable solution into a stable one but also to generate chaotic solutions.

In this article, the nonlinear Duffing oscillator with the time delay loop and in particular the scenario of the transition to a chaotic solution will be examined.

4 A single Duffing oscillator with time delay loop

A single Duffing oscillator with a time delay loop can be represented by a dimensionless differential equation:

$$\ddot{z}(t) + c\dot{z}(t) + az(t) + bz(t)^3 = p\left[z(t-\tau) - z(t)\right] \quad (1)$$

Substituting $x = z$, $y = \dot{z}$, a single second-order differential equation, Eq. (1), can be converted into two first-order differential equations:

$$\dot{x}(t) = y(t) \quad (2)$$
$$\dot{y}(t) = -cy(t) - ax(t) - bx(t)^3 + p\left[x(t-\tau) - x(t)\right]$$

where τ is the time delay value – bifurcation parameter, p = 30.0 – delay gain, and the other dimensionless parameters are: a = 1.0; b = 10.0; c = 0.03162.

In this work, the influence of the change in the value of the time delay τ on the dynamics of the tested system was analyzed. The research was carried out in the MATLAB and OriginPro programs. The obtained results are presented by using a bifurcation diagram (Figure 1), phase portraits, Poincaré maps, and FFT analysis (Figures 2–5). All figures show dimensionless values.

Figure 1 shows a bifurcation diagram where three consecutive Hopf bifurcations (τ_1, τ_2, τ_3) can be observed. For the parameter value τ_1 = 0.01, the first Hopf bifurcation takes place. The system changes from a stationary solution to a periodic solution (Figure 2a, 2b), and the first vibration frequency appears. When analyzing the resulting frequency

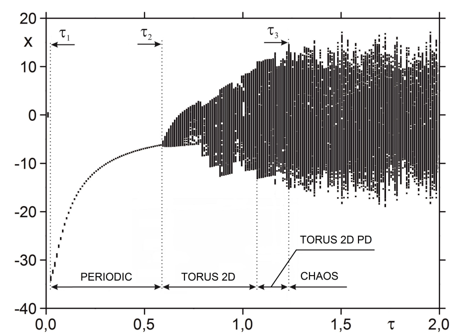

Figure 1: Bifurcation diagram of displacement x versus delay parameter τ for a single Duffing oscillators with time delay loop.

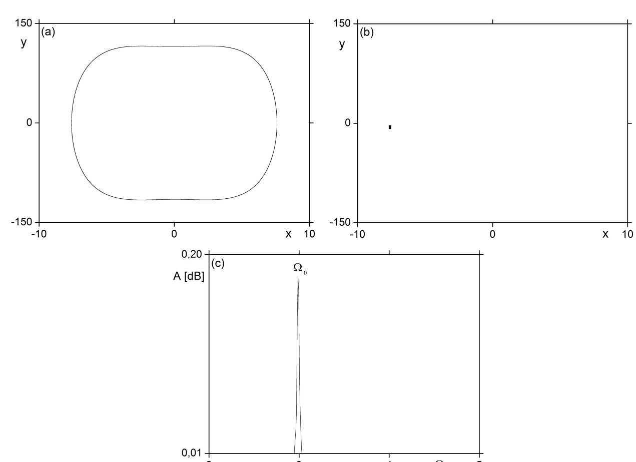

Figure 2: Phase portrait (a), Poincare map (b) and FFT analysis (c) for the time delay τ=0.50.

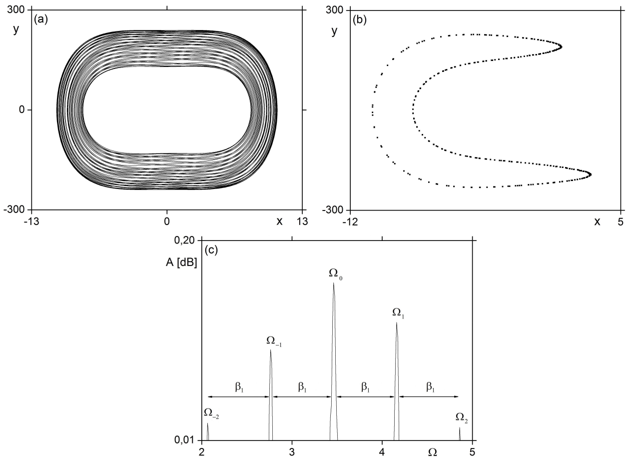

Figure 3: Phase portrait (a), Poincare map (b) and FFT analysis (c) for the time delay τ=0.72.

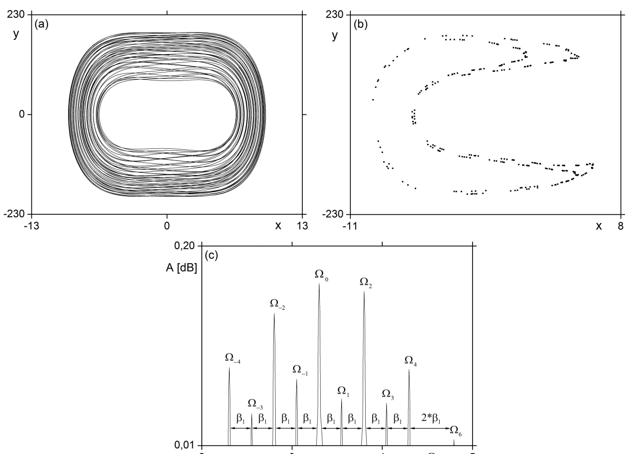

Figure 4: Phase portrait (a), Poincare map (b) and FFT analysis (c) for the time delay τ=1.05.

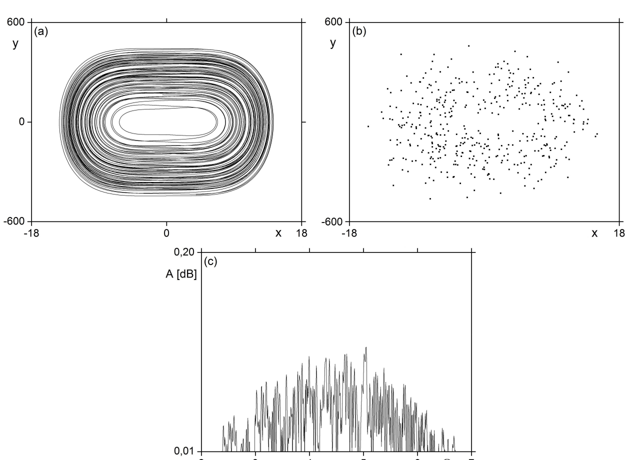

Figure 5: Phase portrait (a), Poincare map (b) and FFT analysis (c) for the time delay τ=1.40.

using the FFT analysis (Figure 2c), this frequency is represented by a single peak Ω_0. The limit cycle in the range of the delay parameter $\tau \in \langle 0.01 \div 0.67 \rangle$ occurs. For $\tau_2 = 0.68$, the second Hopf bifurcation appears. The periodic solution turns into a two-frequency quasi-periodic solution (Figure 3a, 3b). It is represented by a set of points that forms a closed curve on the Poincaré map (Figure 3b). It should be noted that, in Figure 3b, the set of points does not form a closed curve – it is only related to the analysis time. By increasing the analysis time, a closed curve would appear on the map. A 2D torus is formed, which in the FFT analysis (Figure 3c) is represented by two frequencies disproportionate to each other. The first frequency of vibrations is represented by the peak Ω_0, while the second frequency is disproportionate to the first, related to the constant value of the shift of the peaks in relation to the peak Ω_0. The values of newly formed frequencies in relation to the peak Ω_0 can be calculated using the formula:

$$\Omega_n = \Omega_0 + n\beta_1 \tag{3}$$

where β_1 – a constant offset value between peaks and n – analyzed frequency number. For example, the frequency Ω_2 in Figure 3c is:

$$\Omega_2 = \Omega_0 + 2\beta_1 = 3.46133 + 2 * 0.69869 = 4.85871$$

Further increasing the time delay ($\tau \in \langle 1.05 \div 1.17 \rangle$) results in a period-doubling bifurcation on the 2D torus (Figure 4a, 4b). Similar to Figure 3b, in Figure 4b the set of points that form a closed curve was not obtained. This is only related to the analysis time because by increasing the analysis time, a closed curve would appear on the map. Figure 4c shows that between the peaks Ω_{-4}, Ω_{-2}, Ω_0, Ω_2, Ω_4, Ω_6 – representing the 2D torus – there are new ones that divide the distance between them exactly by half. To calculate the value of one of the newly formed peaks, the formula in Eq. (3) should be used. For example, the frequency Ω_{-3} in Figure 4c is:

$$\Omega_{-3} = \Omega_0 - 3\beta_1 = 3.30071 - 3 * 0.24896 = 2.55383$$

For the value of the parameter $\Omega_3 = 1.18$, a chaotic solution occurs (Figure 5a, b). In the FFT analysis, the dominant frequencies cannot be specified. All peaks are located very close to each other and in a chaotic manner, as a result of which the analysis and interpretation of the obtained results are impossible to perform.

5 Conclusions

This article examines a single Duffing oscillator with a time delay loop. The research aimed to check the impact of the time delay value on the nature of the solution, in particular the scenario of transition to a chaotic solution. The obtained results using dynamic tools such as bifurcation diagrams, phase portraits, Poincaré maps, and FFT signal spectrum analysis were presented.

After the research for a single Duffing oscillator with a time delay loop, it can be concluded that the scenario of transition to chaotic behavior is the result of three Hopf bifurcations (Landau-Hopf scenario). As a result of the appearance of successive Hopf bifurcations, the first vibration frequency appears – for the first bifurcation (τ_1); or new, disproportionate vibration frequencies appear – for the second and third bifurcations (τ_2, τ_3).

After the conducted research, the appearance of period-doubling bifurcation on the 2D torus was also observed. This bifurcation was associated with the appearance of new frequencies, dividing the distance between the frequencies representing the 2D torus by exactly half.

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